Midterm 2: Review I

Friday, April 1

Gradients

What is a gradient, and what is it good for? Give sketches and examples including but not limited to contour plots, limits, linear approximation, the Chain Rule, and optimization problems.

Some possibilities:

- It's the direction of steepest ascent of f at any given point
- If it's zero, you have a local minimum, maximum, or saddle point.
- $\nabla f \cdot \mathbf{u} = f_{\mathbf{u}}$ at any point for any unit vector $\mathbf{u}$ (the dot product gives the directional derivative)
- $-\nabla f$ gives the direction of steepest descent (if a ball is on a hill, it will roll in the direction $-\nabla f$)
- In a contour plot of f, the gradient is perpendicular to the level curves at any given point. The larger the gradient, the steeper the ascent, the closer the level curves are.
- $f(x+h) \approx f(x) + \nabla f(x) \cdot h$, so the gradient determines the best linear approximation to f.
- The gradient is the normal vector to the plane tangent to the graph of f at any given point.
- Chain rule: if $\mathbf{r}(t)$ is a vector function of t then $df/dt = \nabla f(\mathbf{r}) \cdot \mathbf{r}'(t)$.
- Lagrange multipliers: If we want to minimize or maximize $f(x)$ given $g(x) = k$, then the level curves of f and g are tangent at an optimal point x, meaning that $\nabla f(x) = \lambda \nabla g(x)$ for some scalar λ.

Second Derivative Test

Given a function $f : \mathbb{R}^2 \to \mathbb{R}$, what is the Second Derivative test? Give all possible outcomes of the test and say what they tell you, providing sketches.

Set $D = f_{xx}f_{yy} - f_{xy}^2$.

- If $D > 0$ and $f_{xx} > 0$ (equivalently, $f_{yy} > 0$) then the function is at a local minimum.
- If $D > 0$ and $f_{xx} < 0$ (eq. $f_{yy} < 0$) then the function is at a local maximum.
- If $D < 0$ then the function is at a saddle point.
- If $D = 0$, the second derivative test is inconclusive.

In linear algebra terms: $f(x+h) \approx f(x) + \nabla f \cdot h + h^T \nabla^2 f h$, so if $\nabla f = 0$ then the second derivative test tells you whether $\nabla^2 f$ has two positive eigenvalues (local min), two negative eigenvalues (local max) or one positive and one negative (saddle), or one or to zero eigenvalues (inconclusive, possibly any of the above).

True or False?

1. If ∇f exists everywhere then f is continuous everywhere. TRUE: if a function is differentiable it must be continuous (but not the other way round!)

2. If $f_x = f_y = 0$ at a point (x, y) then f is differentiable at (x, y). FALSE: it might not even be continuous! (come up with examples)

3. If f is differentiable along every straight line going through a point (x, y) then f is differentiable at (x, y). FALSE: Still might not even be continuous!

4. For any x, $f(x - \nabla f(x)) \leq f(x)$. FALSE: the negative gradient is a descent direction, so what's true is that if $\nabla f \neq 0$ then there exists $\lambda > 0$ (possibly very small) such that $f(x - \lambda \nabla f(x)) < f(x)$.

5. If $f(x, y) = 1$ then $\iint_D f(x, y)\, dA$ is equal to the area of the domain D. TRUE

6. For any $a, b \in \mathbb{R}$ and continuous function f, $\int_{x=0}^{a} \int_{y=0}^{b} f(x, y)\, dy\, dx = \int_{y=0}^{b} \int_{x=0}^{a} f(x, y)\, dx\, dy$. TRUE

7. If $f(x, y) = g(x)h(y)$, then $\iint_D f(x, y)\, dA = \left(\iint_D g(x)\, dA\right) \left(\iint_D h(y)\, dA\right)$. FALSE: you can split the integral as a product of two single-variable integrals if the integral is over a rectangle.

8. If $f_{xx} > 0$ and $f_{yy} > 0$ at a point (x, y) then the point (x, y) is a local minimum of the function f. FALSE: if f_{xy}, f_{yx} are large then it could be a saddle point.

9. If (x, y) is a local minimum of a function f then f is differentiable at (x, y) and $\nabla f(x, y) = 0$. FALSE: say, $f(x, y) = |x| + |y|$.

10. If $\nabla f(x, y) = 0$ then (x, y) is a local minimum or maximum of f. FALSE: it could be a saddle point.

11. If $f_{xx} > 0$ and $f_{yy} < 0$ at a point (x, y) then (x, y) is a saddle point of f. TRUE, IF (x, y) is a critical point. Otherwise it's definitely false.

12. If ∇f is never zero then the minimum and maximum of f on a closed and bounded domain D must occur on the boundary. TRUE, assuming f is continuous and differentiable.

13. If f has a critical point in the interior of a closed and bounded domain D then the minimum and maximum of f on D occur in the interior of D. FALSE. (e.g. $f(x, y) = x^2 - y^2$ on the unit circle)

14. If x is a minimum of f given the constraints $g(x) = h(x) = 0$ then $\nabla f(x) = \lambda \nabla g(x)$ and $\nabla f(x) = \mu \nabla h(x)$ for some scalars λ and μ.

    FALSE: $\nabla f(x) = \lambda \nabla g(x) + \mu \nabla h(x)$ for some λ, μ.