10.2/10.3: Parametric Curves and Polar Coordinates

Friday, January 22

Warmup

1. $\sin(\pi/2) = 1$
2. $\sin(5\pi/4) = -\sqrt{2}/2$
3. $\cos(5\pi/3) = 1/2$
4. $\sin(5\pi/3) = -\sqrt{3}/2$
5. $\sin(2\theta) = 0$
6. $\cos(2\theta) = 1$
7. $\frac{d}{dx}\sin(\cos^2(x)) = \cos(\cos^2(x))2\cos x(-\sin x)$
8. $\frac{d}{dx}x\cos x = \cos x - x\sin x$
9. $\dfrac{d}{dx}\dfrac{x}{\sin x} = \dfrac{1}{\sin x} - \dfrac{x}{\sin^2 x}\cos x$

1. Given (x, y), what is (r, θ)? $r = \sqrt{x^2 + y^2}, \theta = \arctan y/x$
2. Given (r, θ), what is (x, y)? $x = r\cos\theta, y = r\sin\theta$
3. Describe the path: $(x, y) = (-\sin(3t), \cos(3t)), 0 \leq t \leq \pi$. The particle starts at $(0, 1)$ and travels counterclockwise one-and-a-half times around a circle of radius 1.
4. L'Hospital's rule says what? If $f(x) = 0$ and $g(x) = 0$, then the derivative of f/g at $x = 0$ is $\lim_{x\to 0}\frac{f'(x)}{g'(x)}$. This can be useful for deciding whether a tangent line to a point on a parametric curve is horizontal or vertical.

Calculus with Parametric Curves

If $x = e^t, y = te^{-t}$, find dy/dx and d^2y/dx^2, with and without eliminating the parameter. When is the curve concave upward?
Without eliminating the parameter:

$$dy/dx = (te^{-t})/(e^t)' = (e^{-t} - te^{-t})/e^t = e^{-2t}(1 - t).$$

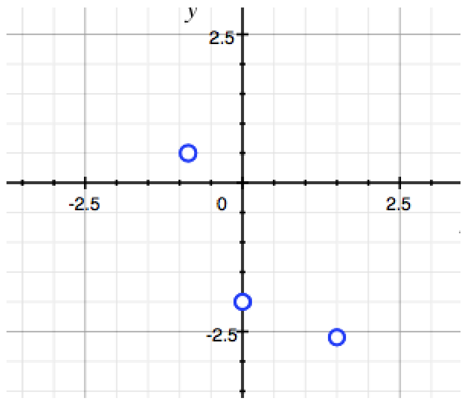
$$d^2y/dx^2 = \frac{(e^{-2t}(1-t))'}{(e^t)'} = \frac{-2e^{-2t}(1-t) - e^{-2t}}{e^t} = (2t - 3)e^{-3t}.$$

Since e^{-3t} is always positive, the curve is concave upward when $t > 3/2$.
With eliminating the parameter: the function $y(t)$ is not invertible, but $x(t)$ is. So get $t = \ln x$, and $y = \ln x/x$. Therefore, $y' = \frac{1-\ln x}{x^2}$ and $y'' = (2\ln x - 3)/x^3$ (note the similarity to the expression for y'' in terms of t).
If $x = 3t^2 + 1$ and $y = t^3 - 1$, at what points on the curve does the tangent line have slope $\frac{1}{2}$?
$dy/dx = 3t^2/6t = t/2$, so the tangent line to the curve has slope $\frac{1}{2}$ when $t = 1$, which is at the point $(4, 0)$.
Find the slope of the tangent line to the trochoid $x = r\theta - d\sin\theta, y = r - d\cos\theta$ in terms of θ. (Here, the particle is distance d from the center of a circle of radius r, rolling on a flat surface.) Find all horizontal and vertical tangents.

Polar coordinates

Plot. Express in Cartesian coordinates and in at least two other ways in polar coordinates:

1. $(2, 3\pi/2)$: $(x, y) = (0, -2)$, $(r, \theta) = (2, -\pi/2) = (-2, \pi/2)$
2. $(3, -\pi/3)$: $(x, y) = (3/2, -3\sqrt{3/2})$, $(r, \theta) = (3, 5\pi/3) = (-3, 2\pi/3)$
3. $(1, 5\pi/6)$: $(x, y = -\sqrt{3}/2, 1/2)$, $(r, \theta) = (1, -7\pi/6) = (-1, -\pi/6)$.



Express in both Cartesian and polar coordinates:

1. A line through the origin that makes an angle of $\pi/6$ with the positive $x-$axis.
   $\tan(\pi/6) = \sqrt{3}/3$, so the line can be expressed in Cartesian coordinates as $y = \frac{\sqrt{3}}{3}x$.
   In polar coordinates, $\theta = \pi/6$ will do.
2. A vertical line through the point $(3, 3)$.
   In Cartesian coordinates, $x = 3$.
   In polar coordinates, $x = 3$ implies that $r\cos\theta = 3$, so $r = 3/\cos\theta = 3\sec\theta$.

Find the slope of the tangent line to the given curve at the point specified:

1. $r = 2\cos\theta, \theta = \pi/3$

$$
\begin{aligned}
dy/dx &= \frac{dy/d\theta}{dx/d\theta} \\
&= \frac{(2\cos\theta \cdot \sin\theta)'}{(2\cos^2\theta)'} \\
&= \frac{(\sin 2\theta)'}{(\cos 2\theta - 1)'} \\
&= \frac{2\cos 2\theta}{-2\sin 2\theta} \\
&= \frac{-1}{-\sqrt{3}} \\
&= 1/\sqrt{3}.
\end{aligned}
$$

2. $r = 1 + \sin 2\theta, \theta = \pi/4$

$$\begin{aligned}
dy/dx &= \frac{((1+\sin 2\theta)\sin\theta)'}{((1+\sin 2\theta)\cos\theta)'} \\
&= \frac{(1+\sin 2\theta)\cos\theta + 2\cos 2\theta \sin\theta}{-(1+\sin 2\theta \sin\theta) + 2\cos 2\theta\cos\theta} \\
&= \frac{2\sqrt{2}2 + 0}{-2\sqrt{2}/2 + 0} \\
&= -1
\end{aligned}$$

3. $r = 1/\theta, \theta = \pi.$

$$\begin{aligned}
dy/dx &= \frac{(\sin\theta/\theta)'}{(\cos\theta/\theta)'} \\
&= \frac{(\theta\cos\theta - \sin\theta)/\theta^2}{(-\theta\sin\theta - \cos\theta)/\theta^2} \\
&= -\pi
\end{aligned}$$