9.2: Direction Fields and Euler's Method

Wednesday, April 8

Warmup

1. Show that every function of the form $x(t) = t^2 + Bt + C$ is a solution to the differential equation $x'' = 2$.

   $(t^2 + Bt + C)'' = (2t + B)' = 2$.

2. Show that every function of the form $y = Ke^{-ct}$ is a solution to the differential equation $y' = -cy$.

   $y' = (Ke^{-ct})' = -Kce^{-ct} = -c(Ke^{-ct}) = -cy$.

3. Show that every function of the form $y = A\sin x + B\cos x$ solves $y'' = -y$.

   $y'' = (A\sin x + B\cos x)'' = (A\cos x - B\sin x)' = (-A\sin x - B\cos x) = -y$.

Say that the positon x of a particle over time is described by $\frac{dx}{dt} = x(x-1)(x-2)$.

1. For what values of x is the particle moving in the positive direction?

   The particle is moving in the positive direction for $0 < x < 1$ and $x > 2$.

2. For what starting values of x will the particle not move at all?

   The particle will not move if $x = 0, 1, 2$.

3. Suppose the particle starts at $x = 0.1$. What will happen to it as $t \to \infty$?

   $x \to 1$.

4. What if the particle starts at $x = 1.5$?

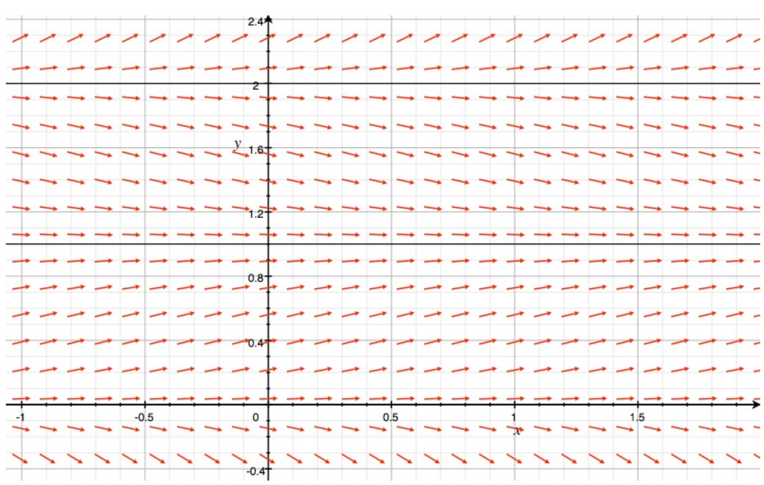
   $x \to 1$.

5. What if the particle starts at $x = 2.5$?

   $x \to \infty$, accelerating the entire time.

Direction Fields

Draw a vector field for the differential equation in the previous question. Use this to sketch some possible paths for the particle in question.



Match each of the following differential equations with the proper vector fields.

1. $y' = y^2$
2. $y' = x - y$
3. $y' = 2 - y$
4. $y' = x$

Particle Simulation/Euler's Method

Consider the solution to the differential equation $y' = y$ with $y(0) = 1$.

1. Estimate $y(1)$ using Euler's Method with a step size of 1.

   $y(1) \approx 2$

2. Estimate $y(1)$ using Euler's Method with a step size of $1/2$.

   $y(1) \approx 2.25$

3. Show that $y = e^x$ is a solution to $y' = y$.

   $y' = (e^x)' = e^x = y$

4. What will happen to your prediction for $y(1)$ as the step size shrinks to zero?

   As the step size Δt goes to zero, the approximation for $y(1)$ will approach e from below.

Air Resistance

Let the velocity of a falling object be described by the equation $\frac{dv}{dt} = -4 - v$, where up is considered to be the positive direction.

1. Sketch the direction field. What will happen to the object's velocity as $t \to \infty$?
2. If the object starts at $x = 20, v = 0$, estimate its position and velocity at $t = 4$ using Euler's Method.

Nullclines

Sketch the direction fields describing the differential equations $y' = y(5 - x - y)$ and $y' = 1 - y - x^2$.

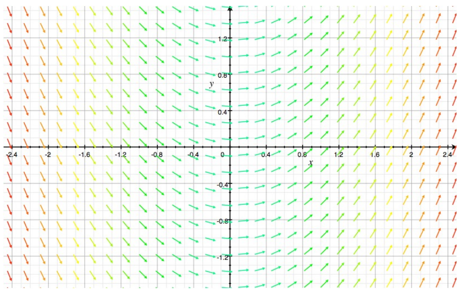


Figure 1: $y' = x$

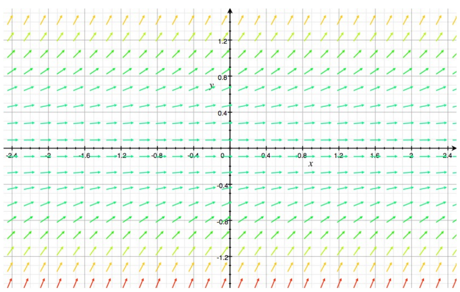


Figure 2: $y' = y^2$

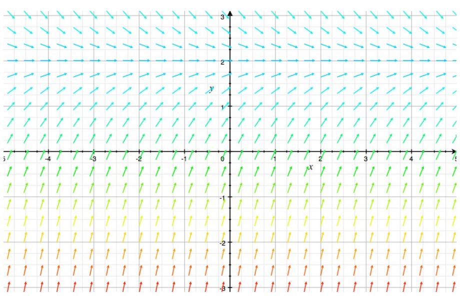


Figure 3: $y' = 2 - y$

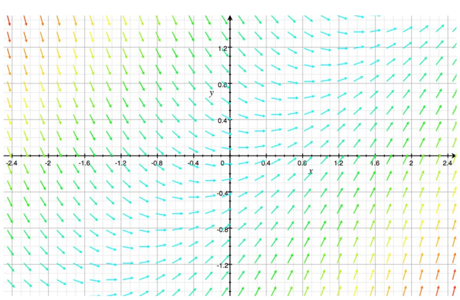


Figure 4: $y' = x - y$