# 17.2: Variation of Parameters

Wednesday, April 22

## Variation of Parameters

To solve $ay'' + by' + cy = G$, for some function $G(x)$:

1. Find linearly independent solutions $y_1, y_2$ to $ay'' + by' + cy = 0$.
2. Set $W = y_1 y_2' - y_2 y_1'$.
3. The solution is
$$y = -y_1 \int \frac{Gy_2}{aW}\,dx + y_2 \int \frac{Gy_1}{aW}\,dx$$

Example: Solve $y'' - 3y' + 2y = e^{-x}$.

1. Since the auxiliary equation is $0 = r^2 - 3r + 2 = (r-2)(r-1)$, two linearly independent solutions to the homogeneous equation are $y_1 = e^x, y_2 = e^{2x}$.
2. $W = y_1 y_2' - y_2 y_1' = e^x(2e^{2x}) - e^{2x}e^x = e^{3x}$.
3. With $a = 1$ and $W$ in the form above, the general solution is
$$\begin{aligned}
y &= -y_1 \int \frac{Gy_2}{aW}\,dx + y_2 \int \frac{Gy_1}{aW}\,dx \\
&= -e^x \int \frac{e^{-x}e^{2x}}{e^{3x}}\,dx + e^{2x} \int \frac{e^{-x}e^{x}}{e^{3x}}\,dx \\
&= -e^x \int e^{-2x}\,dx + e^{2x} \int e^{-3x}\,dx \\
&= -e^x \left(-\frac{1}{2}e^{-2x} + C_1\right) + e^{2x}\left(-\frac{1}{3}e^{-3x} + C_2\right) \\
&= \frac{1}{2}e^{-x} - \frac{1}{3}e^{-x} + C_1 e^x + C_2 e^{2x} \\
&= \frac{1}{6}e^{-x} + C_1 e^x + C_2 e^{2x}
\end{aligned}$$
4. Note that $\frac{1}{6}e^{-x}$ is a solution to $y'' - 3y' + 2y = e^{-x}$ and that for any constants $C_1, C_2$, $C_1 e^x + C_2 e^{2x}$ is a solution to $y'' - 3y + 2y = 0$.
5. Since the right hand side was $e^{-x}$, it would have been simpler to solve this problem using the method of undetermined coefficients (do it). But when the right hand side is more complicated it may be better to use the approach above.

## Exercises

Solve the following differential equations using both the method of undetermined coefficients and variation of parameters:

1. $4y'' + y = \cos x$
2. $y'' - 2y' + y = e^{2x}$
3. $y'' - y' = e^x$

4. $y'' - 2y' - 3y = x + 2$
5. $y'' - 2y' = x + \sin x$
6. $y'' + 5y' + 6y = xe^x$

Solve the following differential equations using variation of parameters:

1. $y'' - 2y' + y = \dfrac{e^x}{x^2}$
2. $y'' + 3y' + 2y = \dfrac{1}{1 + e^x}$

## Power Series Solutions to Differential Equations

. . . have been removed from the syllabus.

END OF NEW MATERIAL