

## 17.2: Variation of Parameters

Wednesday, April 22

### Variation of Parameters

To solve  $ay'' + by' + cy = G$ , for some function  $G(x)$ :

1. Find linearly independent solutions  $y_1, y_2$  to  $ay'' + by' + cy = 0$ .
2. Set  $W = y_1y_2' - y_2y_1'$ .
3. The solution is

$$y = -y_1 \int \frac{Gy_2}{aW} dx + y_2 \int \frac{Gy_1}{aW} dx$$

Example: Solve  $y'' - 3y' + 2y = e^{-x}$ .

1. Since the auxiliary equation is  $0 = r^2 - 3r + 2 = (r - 2)(r - 1)$ , two linearly independent solutions to the homogeneous equation are  $y_1 = e^x, y_2 = e^{2x}$ .
2.  $W = y_1y_2' - y_2y_1' = e^x(2e^{2x}) - e^{2x}e^x = e^{3x}$ .
3. With  $a = 1$  and  $W$  in the form above, the general solution is

$$\begin{aligned} y &= -y_1 \int \frac{Gy_2}{aW} dx + y_2 \int \frac{Gy_1}{aW} dx \\ &= -e^x \int \frac{e^{-x}e^{2x}}{e^{3x}} dx + e^{2x} \int \frac{e^{-x}e^x}{e^{3x}} dx \\ &= -e^x \int e^{-2x} dx + e^{2x} \int e^{-3x} dx \\ &= -e^x \left( -\frac{1}{2}e^{-2x} + C_1 \right) + e^{2x} \left( -\frac{1}{3}e^{-3x} + C_2 \right) \\ &= \frac{1}{2}e^{-x} - \frac{1}{3}e^{-x} + C_1e^x + C_2e^{2x} \\ &= \frac{1}{6}e^{-x} + C_1e^x + C_2e^{2x} \end{aligned}$$

4. Note that  $\frac{1}{6}e^{-x}$  is a solution to  $y'' - 3y' + 2y = e^{-x}$  and that for any constants  $C_1, C_2$ ,  $C_1e^x + C_2e^{2x}$  is a solution to  $y'' - 3y' + 2y = 0$ .
5. Since the right hand side was  $e^{-x}$ , it would have been simpler to solve this problem using the method of undetermined coefficients (do it). But when the right hand side is more complicated it may be better to use the approach above.

### Exercises

Solve the following differential equations using both the method of undetermined coefficients and variation of parameters:

1.  $4y'' + y = \cos x$

Undetermined coefficients:

$$\begin{aligned}y &= A \sin x + B \cos x \\y'' &= -A \sin x - B \cos x \\4y'' + y &= -3(A \sin x + B \cos x) \\A &= 0 \\B &= -1/3 \\y &= -\frac{1}{3} \cos x\end{aligned}$$

The auxiliary equation for  $4y'' + y = 0$  is  $4r^2 + 1 = 0$  with solutions  $r = \pm i/2$ , and so the general solution is  $y = -\frac{1}{3} \cos x + C_1 \cos(x/2) + C_2 \sin(x/2)$ .

Variation of parameters:

- (a)  $y_1, y_2 = \cos(x/2), \sin(x/2)$
- (b)  $W = y_1 y_2' - y_2 y_1' = \cos^2(x/2) + \sin^2(x/2) = 1$ .
- (c)

$$\begin{aligned}y &= -\cos(x/2) \int \frac{\cos x \sin(x/2)}{1} + \sin(x/2) \int \frac{\cos x \cos(x/2)}{1} \\&= -\cos(x/2) \int \cos x \sin(x/2) + \sin(x/2) \int \cos x \cos(x/2)\end{aligned}$$

The relevant trig substitutions to get the solution from here are on p.476 of Stewart.

2.  $y'' - 2y' + y = e^{2x}$

Undetermined coefficients:

$$\begin{aligned}y &= ke^{2x} \\y'' - 2y' + y &= 4ke^{2x} - 4ke^{2x} + ke^{2x} \\&= ke^{2x} \\k &= 1 \\y &= e^{2x}\end{aligned}$$

The auxiliary equation is  $r^2 - 2r + 1$ , with repeated root  $r = 1$ , and so the general solution is  $y = e^{2x} + C_1 e^x + C_2 x e^x$ .

Variation of parameters:

- (a)  $y_1, y_2 = e^x, x e^x$
- (b)  $W = y_1 y_2' - y_2 y_1' = e^{2x}$  (Note: try “Wronskian of  $e^x, x e^x$ ” on WolframAlpha to check this answer)
- (c)

$$\begin{aligned}y &= -e^x \int \frac{e^{2x} x e^x}{e^{2x}} + x e^x \int \frac{e^{2x} e^x}{e^{2x}} \\&= -e^x \int x e^x + x e^x \int e^x \\&= -e^x (x e^x - e^x + C_1) + x e^x (e^x + C_2) \\&= e^{2x} + C_1 e^x + C_2 x e^x\end{aligned}$$

3.  $y'' - y' = e^x$

Undetermined coefficients. . . guessing  $y = ke^x$  will give  $y'' - y' = 0$ , so guess  $y = kxe^x$  instead:

$$\begin{aligned} y &= kxe^x \\ y' &= kxe^x + ke^x \\ y'' &= kxe^x + 2ke^x \\ y'' - y' &= ke^x \\ k &= 1 \\ y &= xe^x \end{aligned}$$

The auxiliary equation is  $r^2 - r = 0$  with solutions  $r = 0, 1$ , so the general solution is  $y = xe^x + C_1 + C_2e^x$ .

4.  $y'' - 2y' - 3y = x + 2$

Undetermined coefficients:

$$\begin{aligned} y &= Ax^2 + Bx + C \\ y' &= 2Ax + B \\ y'' &= 2A \\ y'' - 2y' - 3y &= 2A - 4Ax - 2B - 3Ax^2 - 3Bx - 3C \\ -3Ax^2 + (-3B - 4A)x + (2A - 2B - 3C) &= x + 2 \\ A &= 0 \\ B &= -1/3 \\ C &= -4/9 \\ y &= -x/3 - 4/9 \end{aligned}$$

The auxiliary equation is  $r^2 - 2r - 3 = 0$  with solutions  $r = -1, 3$ , so the general solution is  $y = -x/3 - 4/9 + C_1e^{-x} + C_2e^{3x}$ .

5.  $y'' - 2y' = x + \sin x$

For undetermined coefficients: first guess  $y = Ax^2 + Bx$  (+C, but the constant coefficient will disappear when taking derivatives) to solve  $y'' - 2y' = x$ :

$$\begin{aligned} y &= Ax^2 + Bx \\ y' &= 2Ax + B \\ y'' &= 2A \\ y'' - 2y' &= 2(A - B) - 2Ax \\ 2(A - B) - 2Ax &= x \\ A &= -1/2 \\ B &= -1/2 \\ y &= -x^2/2 - x/2 \end{aligned}$$

Then do the same for  $\sin x$ , guessing

$$\begin{aligned}y &= A \sin x + B \cos x \\y' &= A \cos x - B \sin x \\y'' &= -A \sin x - B \cos x \\y'' - 2y' &= (2B - A) \sin x - (B + 2A) \cos x \\&= \sin x \\A &= -1/5 \\B &= 2/5 \\y &= -\sin(x)/5 + 2 \cos(x)/5\end{aligned}$$

A particular solution to the equation is therefore  $y = -x^2/2 - x/2 - \sin(x)/5 + 2 \cos(x)/5$ . The auxiliary equation for the homogeneous case is  $r^2 - 2r = 0$  with solutions  $r = 0, 2$ , so the general solution is

$$y = C_1 + C_2 e^{2x} - x^2/2 - x/2 - \sin(x)/5 + 2 \cos(x)/5$$

6.  $y'' + 5y' + 6y = xe^x$  (EDIT: original problem said 6 instead of  $6y$ )

Undetermined coefficients:

$$\begin{aligned}y &= (Ax + B)e^x \\y' &= (Ax + A + B)e^x \\y'' &= (Ax + 2A + B)e^x \\y'' + 5y' + 6y &= 12Ax e^x + (7A + 12B)e^x \\A &= 1/12 \\B &= -7/144\end{aligned}$$

The auxiliary equation to the homogeneous case is  $r^2 + 5r + 6 = 0$  with solutions  $r = -2, -3$ , so the general solution is  $y = c_1 e^{-3x} + c_2 e^{-2x} + xe^x/12 - 7e^x/144$ .

Solve the following differential equations using variation of parameters:

1.  $y'' - 2y' + y = \frac{e^x}{x^2}$

(a)  $y_1 = e^x, y_2 = xe^x$

(b)  $W = y_1 y_2' - y_2 y_1' = e^{2x}$

(c)

$$\begin{aligned}y &= -e^x \int \frac{e^x x e^x}{x^2 e^{2x}} + x e^x \int \frac{e^x e^x}{x^2 e^{2x}} \\&= -e^x \int \frac{1}{x} + x e^x \int \frac{1}{x^2} \\&= -e^x (\ln x + C_1) + x e^x (-1/x + C_2) \\&= -e^x \ln x - e^x + C_1 e^x + C_2 x e^x\end{aligned}$$

2.  $y'' + 3y' + 2y = \frac{1}{1 + e^x}$  (EDIT: originally had 2 instead of  $2y$ )

(a)  $y_1 = e^{-x}, y_2 = e^{-2x}$

(b)  $W = -e^{-3x}$

(c)

$$\begin{aligned}y &= -e^{-x} \int \frac{e^{-2x}}{(1+e^x)(-e^{-3x})} + e^{-2x} \int \frac{e^{-x}}{(1+e^x)(-e^{-3x})} \\&= e^{-x} \int \frac{e^x}{1+e^x} - e^{-2x} \int \frac{e^{2x}}{1+e^x} \\&= e^{-x}(\ln(e^x+1) + C_1) - e^{-2x}(e^x - \ln(e^x+1) + C_2) \\&= C_1 e^{-x} + C_2 e^{-2x} + e^{-x} \ln(1+e^x) + e^{-2x} \ln(1+e^x)\end{aligned}$$

## Power Series Solutions to Differential Equations

... have been removed from the syllabus.

END OF NEW MATERIAL