Trigonometric Substitutions

Wednesday, January 28

Speed Round

Simplify the following expressions:

1. $\tan^2(x) + 1$

   Answer: $\sec^2(x)$

2. $\sin^2(x) + \cos^2(x)$

   Answer: 1

3. $\sec^2(x) - 1$

   Answer: $\tan^2(x)$

4. $\sqrt{1 - \sin^2(x)}$

   Answer: $\cos(x)$ (or $|\cos(x)|$)

5. $\sqrt{1 + \tan^2(x)}$

   Answer: $\sec(x)$ (or $|\sec(x)|$)

6. $\sqrt{a^2 - a^2\cos^2(x)}$

   Answer: $a\sin(x)$ (or $|a\sin(x)|$)

7. $\sqrt{(a\sec(x))^2 - a^2}$

   Answer: $a\tan(x)$ (or $|a\tan(x)|$)

8. $1 - \cos^2(x)$

   Answer: $\sin^2(x)$

Decide which substitution would be most appropriate for evaluating each of the following integrals. Note: it will not always be a trig substitution.

1. $\int x\sqrt{1+x^2}\,dx$

   Answer: $u = 1 + x^2$

2. $\int \frac{1}{\sqrt{1-x^2}}\,dx$

   Answer: $x = \sin\theta$ (or $x = \cos\theta$)

3. $\int \frac{1}{(x^2+1)^2}\,dx$

   Answer: $x = \tan\theta$

4. $\int \frac{1}{\sqrt{x^2-4}}\,dx$

   Answer: $x = 2\sec\theta$

5. $\int \frac{x}{\sqrt{1+x^2}}\,dx$

   Answer: $u = 1 + x^2$

6. $\int \frac{x^2}{\sqrt{4x^2-9}}\,dx$

   Answer: $2x = 3\sec\theta$

7. $\int \frac{\sqrt{x^2-1}}{x}\,dx$

   Answer: $x = \sec\theta$

8. $\int \frac{\sqrt{1+x^2}}{x}\,dx$

   Answer: $x = \tan\theta$

9. $\int x^3\sqrt{1-x^2}\,dx$

   Answer: $u = x^2$

10. $\int \frac{1}{\sqrt{9-25x^2}}\,dx$

    Answer: $5x = 3\sin\theta$

11. $\int \frac{x}{x^2-9}\,dx$

    Answer: $u = x^2 - 9$

12. $\int \frac{x^2}{\sqrt{x^2-9}}\,dx$

    Answer: $x = 3\sec\theta$

Theorem: $\bigcirc = \pi r^2$

Prove (finally!) that the area of a circle of radius r is πr^2.

$$\begin{aligned}
\bigcirc &= 2\int_{-r}^{r} \sqrt{1r^2 - x^2}\, dx \\
&= 2\int_{x=-r}^{r} \sqrt{r^2 - r^2\sin^2\theta} r\cos\theta\, d\theta \\
&= 2r^2\int_{x=-r}^{r} \cos^2\theta\, d\theta \\
&= r^2\int_{x=-r}^{r} 1+\cos(2\theta)\, d\theta \\
&= r^2\left(\theta + \frac{1}{2}\sin(2\theta)|_{x=-r}^{r}\right) \\
&= r^2\left(\arcsin(x) + x\sqrt{1-x^2}|_{x=-r}^{r}\right) \\
&= \pi r^2
\end{aligned}$$

Alternately, change the bounds of the integral appropriately:

$$\begin{aligned}
r^2\left(\theta + \frac{1}{2}\sin(2\theta)|_{x=-r}^{r}\right) &= r^2\left(\theta + \frac{1}{2}\sin(2\theta)|_{\theta=-\pi/2}^{\pi/2}\right) \\
&= \pi r^2
\end{aligned}$$

More Trig Identities

Simplify the following expressions:

1. $\sin(\arccos x)$
   Answer: $\sqrt{1-x^2}$
2. $\cos(\arcsin x)$
   Answer: $\sqrt{1-x^2}$
3. $\sec(\arctan x)$
   Answer: $\sqrt{x^2+1}$
4. $\cos(\arctan x)$
   Answer: $\frac{1}{\sqrt{1+x^2}}$
5. $\tan(\arcsin x)$
   Answer: $x/\sqrt{1-x^2}$
6. $\tan(\sec^{-1} x)$
   Answer: $\sqrt{x^2-1}$

1. If $t = \sec\theta$, then what is $\tan\theta$ in terms of t?
   Answer: $\sqrt{t^2-1}$
2. If $t = \sin\theta$, then what is $\cos\theta$ in terms of t?
   Answer: $\sqrt{1-t^2}$
3. If $t = \tan\theta$, then what is $\sec\theta$ in terms of t?
   Answer: $\sqrt{t^2+1}$

Completing the square

Put each of the following quadratics in the form $(Ax+B)^2+C$, then decide which trig substitution would be appropriate:

1. $\sqrt{x^2+4x+8}$
   Answer: $(x+2)^2+4$, sub $x+2 = 2\tan\theta$
2. $\sqrt{-x^2-6x}$
   Answer: $9-(x+3)^2$, sub $x+3 = 3\sin\theta$
3. $\sqrt{x^2-2x+10}$
   Answer: $(x-1)^2+9$, sub $x-1 = 3\tan\theta$
4. $\sqrt{-4x^2-12x-5}$
   Answer: $4-(2x+3)^2$, sub $2x+3 = 2\sin\theta$
5. $\sqrt{9x^2-6x-3}$
   Answer: $(3x-1)^2-4$, sub $3x-1 = 2\sec\theta$
6. $\sqrt{4x^2-4x+2}$
   Answer: $(2x-1)^2+1$, sub $2x-1 = \tan\theta$
7. $\sqrt{25x^2+10x+10}$
   Answer: $(5x+1)^2+9$, sub $5x+1 = 3\tan\theta$
8. $\sqrt{-16x^2+24x-5}$
   Answer: $4-(4x-3)^2$, sub $4x+3 = 2\sin\theta$

Bonus: Prove the quadratic formula.

$$
\begin{aligned}
Ax^2 + Bx + C &= 0 \\
x^2 + (B/A)x + (C/A) &= 0 \\
(x + \frac{B}{2A})^2 - \frac{B^2}{4A^2} + C/A &= 0 \\
4A^2(x + \frac{B}{2A})^2 - B^2 + 4AC &= 0 \\
4A^2(x + \frac{B}{2A})^2 &= B^2 - 4AC \\
2A(x + \frac{B}{2A}) &= \pm\sqrt{B^2 - 4AC} \\
x + \frac{B}{2A} &= \frac{\pm\sqrt{B^2 - 4AC}}{2A} \\
x &= \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}
\end{aligned}
$$

Weierstrass substitution

$$t = \tan(x/2);\ \sin(x) = \frac{2t}{1+t^2};\ \cos(x) = \frac{1-t^2}{1+t^2};\ dx = \frac{2}{1+t^2}dt$$

Integrate:

1. $\int \frac{1}{\sin x}\, dx$

$$
\begin{aligned}
\int \frac{1}{\sin x}\, dx &= \int \frac{1+t^2}{2t}\frac{2}{1+t^2}\, dt \\
&= \ln(t) \\
&= \ln|\tan(x/2)|
\end{aligned}
$$

2. $\int \frac{1}{\cos x}\, dx$

$$
\begin{aligned}
\int \frac{1}{\cos x}\, dx &= \int \frac{1+t^2}{1-t^2}\frac{2}{1+t^2}\, dt \\
&= \int \frac{2}{(1+t)(1-t)}\, dt \\
&= \int \frac{1}{1+t} + \frac{1}{1-t}\, dt \\
&= \ln(1+t) + \ln(1-t) \qquad = \ln|1+\tan(x/2)| + \ln|1-\tan(x/2)|
\end{aligned}
$$

3. $\int \frac{1}{1+\sin x}\, dx$

$$
\begin{aligned}
\int \frac{1}{1+\sin x}\,dx &= \int \frac{1+t^2}{t^2+2t+1}\frac{2}{1+t^2}\,dt \\
&= \int \frac{2}{(1+t)^2}\,dt \\
&= \frac{-2}{t+1} \\
&= \frac{-2}{1+\tan(x/2)}
\end{aligned}
$$

4. $\int \frac{1}{1+\cos x}\,dx$

$$
\begin{aligned}
\int \frac{1}{1+\cos x}\,dx &= \int dt \\
&= t \\
&= \tan(x/2)
\end{aligned}
$$