Trigonometric Integrals–Solutions

Friday, January 23

Review

Compute the following integrals using integration by parts. It might be helpful to make a substitution.

1. $\int_1^{e^2} \sqrt{x}\ln(x)\,dx$
   $\frac{4}{9}(1+2e^3)$

2. $\int_0^1 x\sqrt{1+x}\,dx$
   $\frac{4}{15}(1+\sqrt{2})$

Discuss: does the best strategy for solving each of the following integrals use substitution, integration by parts, both, or neither?

1. $\int x\ln(x)\,dx$: IBP $(u = \ln x)$
2. $\int \frac{\ln(x)}{x}\,dx$: sub $u = \ln x$
3. $\int \frac{1}{x\ln(x)}\,dx$: sub $u = \ln x$
4. $\int 1/x\,dx$: neither
5. $\int \ln(x)\,dx$: IBP $(u = \ln x)$
6. $\int \cos(x)e^{\sin(x)}\,dx$: sub $u = \sin x$
7. $\int x\sqrt{1+x}\,dx$: IBP $u = x$
8. $\int x\sqrt{x}\,dx$: niether (rewrite as $x^{3/2}$)
9. $\int \sin(x)\cos(x)e^{\sin(x)}\,dx$: sub $u = \sin x$

Trig Formulas to Memorize:

1. $\sin^2(x) + \cos^2(x) = 1$
2. $\sin(2x) = 2\sin(x)\cos(x)$
3. $\cos(2x) = \cos^2(x) - \sin^2(x)$
4. $\tan^2(x) + 1 = \sec^2(x)$.
5. $\int \sin(x)\,dx = -\cos(x) + C$
6. $\int \cos(x)\,dx = \sin(x) + C$
7. $\int \sec^2(x)\,dx = \tan(x) + C$
8. $\int \sec(x)\tan(x)\,dx = \sec(x) + C$

Also Good to Know:

1. $\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$
2. $\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$
3. $\cos(2x) = 2\cos^2(x) - 1$
4. $\cos(2x) = 1 - 2\sin^2(x)$
5. $\sin^2(x) = (1 - \cos(2x))/2$
6. $\cos^2(x) = (1 + \cos(2x))/2$

Formulas to Write on a Cheat Sheet:

Everything else.

Speed Round

1. $\int \cos(x)\, dx$: $\sin x$
2. $\int \sin(x)\, dx$: $-\cos x$
3. $\sin^2(x) + \cos^2(x)$: 1
4. $\sqrt{1 - \cos^2(x)}$: $\sin x$
5. $(a + b)(a - b)$: $a^2 - b^2$
6. $\int \sec^2(x)\, dx$: $\tan x$
7. $(1 + \cos(x))(1 - \cos(x))$: $\sin^2 x$
8. $\cos^4(x) - \sin^4(x)$: $(\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x) = \cos^2 x - \sin^2 x = \cos 2x$
9. $(1 - x^2)/(1 - x)$: 1+x
10. $\cos^2(x)/(1 - \sin(x))$: $1 + \sin x$
11. $\sqrt{1 - \sin^2(x)}$: $\cos x$
12. $\frac{d}{dx} \tan(x)$: $\sec^2 x$
13. $\frac{d}{dx} \sec(x)$. $\sec x \tan x$
14. $\sec^2(x) - 1$: $\tan x$
15. $\cos(2x) + 1$: $2\cos^2 x - 1 + 1 = 2\cos^2 x$

Identities

Prove the following trig identities using only $\cos^2(x) + \sin^2(x) = 1$ and sine and cosine addition formulas:

1. $\tan^2(x) + 1 = \sec^2(x)$

$$\begin{aligned} \tan^2(x) + 1 &= \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} \\ &= \frac{\sin^2 x + \cos^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x \end{aligned}$$

2. $\sin^2(x) = (1 - \cos(2x))/2$

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ \cos 2x &= 1 - 2\sin^2 x \\ 1 - \cos 2x &= 2\sin^2 x \\ (1 - \cos 2x)/2 &= \sin^2 x \end{aligned}$$

3. $\cos^2(x) = (1+\cos(2x))/2$

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ \cos 2x &= 2\cos^2 x - 1 \\ 1+\cos 2x &= 2\cos^2 x \\ (1+\cos 2x)/2 &= \cos^2 x \end{aligned}$$

4. $\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$

$$\begin{aligned} \cos(a-b) - \cos(a+b) &= \cos a \cos b + \sin a \sin b - (\cos a \cos b - \sin a \sin b) \\ \cos(a-b) - \cos(a+b) &= 2\sin a \sin b \\ \frac{1}{2}[\cos(a-b) - \cos(a+b)] &= \sin a \sin b \end{aligned}$$

Integrals

Evaluate the following integrals:

1. $\int \sin^2(\sqrt{x})/\sqrt{x}\, dx$
   sub $u = \sqrt{x}$, then use the cosine substitution for $\sin^2 u$ to get $\sqrt{x} - \sin(2\sqrt{x})/2$
2. $\int \sqrt{1+\cos(2x)}\, dx$
   Use $1 + \cos 2x = 2\cos^2 x$ to turn the integral into $\int \sqrt{2}\sin x = -\sqrt{2}\cos x$
3. $\int \frac{1}{1+\sin(x)}\, dx$
   Multiply by $\frac{1-\sin x}{1-\sin x}$ to get $\int \frac{1-\sin x}{\cos^2 x} = \int \sec^2 x - \int \sec x \tan x = \tan x - \sec x$
4. $\int \tan(x)\, dx$: $-\ln(\cos x)$
5. $\int \tan^2(x)\, dx$: Use $\tan^2 = \sec^2 -1$, and get $\tan x - x$
6. $\int \tan^3(x)\, dx$: sub $\tan^2 = \sec^2 -1$, eventually get $\sec^2(x)/2 + \log(\cos x)$.
7. $\int \sqrt{1-\cos(4x)}\, dx$: Sub $\sqrt{1-\cos(4x)} = \sqrt{2\sin^2(2x)} = \sqrt{2}\sin(2x)$. For the integral, get $-\frac{\sqrt{2}}{2}\cos(2x)$.
8. $\int \frac{1}{\cos(x)-1}\, dx$
   Multiply by $\frac{1+\cos x}{1-\cos x}$.

Bonus

1. Show that $\frac{1}{2}\ln\left(\frac{1+\sin(x)}{1-\sin(x)}\right) + C = \ln(\sec(x) + \tan(x)) + C$.
2. Evaluate $\int_0^{2\pi} \sin(3x)\sin(5x)\sin(7x)\, dx$
   The integral is zero.