CONVERGENCE TEST FOR SERIES

Test	Series Form	Conditions for Convergence	Conditions for Divergence	Samples
Divergence or n^{th} term test	$\sum_{n=1}^{\infty} a_n$	CAN'T BE USED	$\lim_{n\to\infty} a_n \neq 0$	$\sum_{n=1}^{\infty} \frac{n+1}{5n}$
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	$\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^n$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	$\sum_{n=1}^{\infty} \frac{1}{n^2}$
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \leq a_n$ $\lim_{n\to\infty} a_n = 0$	See divergence test	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$
Integral	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)dx$ converges	$\int_1^{\infty} f(x)dx$ diverges	$\sum_{n=1}^{\infty} \frac{\ln n}{n}$
Comparison	$\sum_{n=1}^{\infty} a_n$	if $0 < a_n < b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	if $0 < b_n < a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	$\sum_{n=1}^{\infty} \frac{1}{n^2+5}$
Limit Comparison	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	$\sum_{n=1}^{\infty} \frac{1}{3n^2-2n}$
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert > 1$	$\sum_{n=1}^{\infty} \frac{2^n}{n!}$ Note: Inconclusive if limit = 1
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$	$\sum_{n=1}^{\infty} \left(\frac{4n}{5n-3}\right)^n$ Note: Inconclusive if limit = 1

Samples:

1. $\sum_{n=1}^{\infty} \frac{n+1}{5n}$ Diverges because $\lim_{n\to\infty} \frac{n+1}{5n} = \frac{1}{5}$ since the $\lim_{n\to\infty} a_n \neq 0$, the series diverges by the n^{th} term (or divergence) test.

2. $\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^n$ is a geometric series with $r = \frac{2}{3}$ since $|r| < 1$, the series converges to $\frac{\frac{10}{3}}{1-\frac{2}{3}} = 10$

   It can also be shown by ratio test. $\lim_{n\to\infty} \frac{5\left(\frac{2}{3}\right)^{n+1}}{5\left(\frac{2}{3}\right)^n} = \lim_{n\to\infty} \frac{\frac{2}{3}\left(5\left(\frac{2}{3}\right)^n\right)}{5\left(\frac{2}{3}\right)^n} = \frac{2}{3}$ since the limit < 1, it converges.

3. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the *p*-test since $p > 1$.

   It can also be shown to converges by the integral test since $\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{a\to\infty}\left(\frac{-1}{a} + 1\right) = 1$ since the integral converges the series converges.

4. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ The series is alternating and $\lim_{n\to\infty} \frac{1}{n} = 0$ The series converges by the alternating series test since the terms decreases and the limit of a_n goes to 0.

5. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges by the integral test because $\int_1^{\infty} \frac{\ln x}{x}dx = \lim_{a\to\infty}\left(\frac{\ln^2 a}{2} - 0\right) = \infty$

6. $\sum_{n=1}^{\infty} \frac{1}{n^2+5}$ converges since each term $\frac{1}{n^2+5} < \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the comparison test.

7. $\sum_{n=1}^{\infty} \frac{1}{3n^2-2n}$ is comparable to $\sum_{n=1}^{\infty} \frac{1}{n^2}$, using the limit comparison test, look at the limit of the ratio $\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{3n^2-2n}} = \lim_{n\to\infty} \frac{3n^2-2n}{n^2} = 3$ Since the limit is not zero and exist, both the series converge or diverge. Given that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty} \frac{1}{3n^2-2n}$ also converges.

8. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ Look at the ratio $\lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}\right| = \lim_{n\to\infty}\left|\frac{2}{n+1}\right| = 0$. Since the limit is less than 1, the series converges by the ratio test.

9. $\sum_{n=1}^{\infty} \left(\frac{4n}{5n-3}\right)^n$ Using the root test $\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{4n}{5n-3}\right)^n\right|} = \lim_{n\to\infty}\left|\frac{4n}{5n-3}\right| = \frac{4}{5}$. Since the limit is less than 1, the series converges.