Sequences
Wednesday, February 18

Review
1. \( \ln(a \cdot b) = \)
2. \( \ln(x^a) = \)
3. \( x^y = e^? \)
4. \( 5! = \)
5. \( e^x = 1 + x + \ldots \)
6. \( (a + b)(a - b) = \)
7. \( \ln(n + 1) - \ln(n) = \ln(?) \)
8. \( \sqrt{n^2 + n} - n = ? \)
9. \( \lim_{x \to 0} \frac{\sin x}{x} \)

Sequences
Write the first four or five terms (starting from \( n = 0 \)) of each of the following sequences:
1. \( a_n = 6 \cdot \left( \frac{2}{3} \right)^n \)
2. \( a_n = (−1)^n/(2n + 1) \)
3. \( a_n = n! - 3^n \)
Find a formula that produces each of the following sequences:
1. \( \{1, −1, 1, −1, 1, \ldots\} \)
2. \( \{−1, 4, −9, 16, −25, \ldots\} \)
3. \( \{4, 2, 1/2, 1/4, \ldots\} \)

The Hierarchy of Growth
Order the following sequences from smallest to largest as \( n \to \infty \):
\[ n!, \ln(\ln(n)), 3n + 5, 7, n^{0.0001}, 0.8^n, e^n/200, n^6, \sqrt{n}, \sqrt{9n^2 + 3n + 2}, \ln(n), 1.01^n, n^{100} - 2 \]

Determining Convergence of a Sequence
Determine whether each of the following sequences has a limit of 0 or \( \infty \):
1. \( e^n/n! \)
2. \( n^2/\ln(n) \)
3. \( \sqrt{n}/1.01^n \)
4. \( \ln(n)/\ln(\ln(n)) \)
5. \( e^n/n^e \)
6. \( n^2/\sqrt{n^5 + 2} \)
7. \( \frac{n^5 + 10 \ln(n)}{5n^5 + 1.01^n} \)
8. \( \frac{3n + 5 + \ln \ln n}{n!} \)
9. \( \frac{1.05^n + n^3}{1.04^n + \sqrt{n}} \)
10. \( \frac{\sqrt{n^5 + 1}(n + 1)^3}{(n + \ln n)^3} \)
11. \( \frac{e^n + e^{-n}}{2n} \)
12. \( \frac{n \ln \ln n}{1 + \ln n} \)
13. \( \frac{e^n \ln n + \sqrt{n}}{n!} \)
14. \( \frac{1.01^n}{n^{1.01}} \)
Each of the following sequences has a positive finite limit. Find the limit:

1. $\sqrt{n^2 + 2n} - n$
2. $\sqrt{n^2 + 5n + 1} - n$
3. $\sqrt{4n^2 + 6n + 3} - 2n$
4. $n \ln(1 + 1/n)$
5. $n \sin(1/n)$
6. $(1 + 1/n)^n$
7. $n(\ln(n + 1) - \ln(n))$
8. $n\sqrt{n^2 + 1} - n^2$
9. $n(e^{1/n} - 1)$

**True or False?**

If true, explain your reasoning. If false, find a counterexample.

1. If $\{a_n\}$ converges, then $\lim_{n \to \infty} a_n = 0$.
2. If $\{a_n\}$ diverges, then $\lim_{n \to \infty} a_n \neq 0$.
3. If $\{a_n\}$ converges, then there is an $N$ such that $n \geq N$ implies $|a_n - a_{n+1}| < N$.
4. If $\{a_n\}$ diverges, then for any $\epsilon > 0$ there is an $N$ such that $n \geq N$ implies $|\lim_{n \to \infty} a_n - a_n| > \epsilon$.
5. If $\lim_{n \to \infty} a_n = 0$ but $\{b_n\}$ diverges, then the sequence $\{a_n b_n\}$ diverges.
6. If $\{a_n\}$ and $\{b_n\}$ both converge, then $\{a_n + b_n\}$ converges.
7. If $\{a_n\}$ and $\{b_n\}$ both converge, then $\{a_n b_n\}$ converges.
8. If $\{a_n\}$ converges, then $\lim_{n \to \infty} a_n/n = 0$.
9. For every $\{a_n\}$ there is some $\{b_n\}$ such that $\{a_n b_n\}$ diverges.
10. Every bounded sequence is convergent.
11. Every bounded convergent sequence is monotonic.
12. If $f$ is a function and $\lim_{n \to \infty} a_n = L$, then $f(L) = \lim_{n \to \infty} f(a_n)$.
13. If $f$ is a continuous function and $\lim_{n \to \infty} a_n = L$, then $f(L) = \lim_{n \to \infty} f(a_n)$.
14. If $\{a_n\}$ converges, then $\{a_n^2\}$ converges.

**Bonus**

1. Prove that $\lim_{n \to \infty} \ln(n)/n^\epsilon = 0$ for any $\epsilon > 0$.
2. Prove that $\lim_{n \to \infty} k^n/n! = 0$ for any constant $k > 0$. 

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