Sequences

Wednesday, Febrary 18

Review

1. $\ln(a \cdot b) = \ln a + \ln b$
2. $\ln(x^a) = a \ln x$
3. $x^y = e^{x \ln y}$
4. $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$
5. $e^x = 1+x+x^2/2!+x^3/3!+\ldots$
6. $(a+b)(a-b) = a^2 - b^2$
7. $\ln(n+1)-\ln(n) = \ln(1+1/n)$
8. $\sqrt{n^2+n}-n = \dfrac{n}{n+\sqrt{n^2+n}}$
9. $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$

Sequences

Write the first four or five terms (starting from $n = 0$) of each of the following sequences:

1. $a_n = 6 \cdot \left(\frac{2}{3}\right)^n$

$$a_0 = 6, a_1 = 4, a_2 = 8/3, a_3 = 16/9, a_4 = 32/27$$

2. $a_n = (-1)^n/(2n+1)$

$$a_0 = 1, a_1 = -1/3, a_2 = 1/5, a_3 = -1/7$$

3. $a_n = n! - 3^n$

$$a_0 = 0, a_1 = -2, a_2 = -7, a_3 = -21, a_4 = -57, a_5 = -123, a_6 = -9, a_7 = 2853$$

Find a formula that produces each of the following sequences (starting from $n = 1$):

1. $\{1, -1, 1, -1, 1, \ldots\}$

$$a_n = (-1)^{n+1}$$

2. $\{-1, 4, -9, 16, -25, \ldots\}$

$$a_n = (-1)^n/n^2$$

3. $\{4, 2, 1, 1/2, 1/4, \ldots\}$

$$a_n = 8/2^n = 2^{3-n}$$

The Hierarchy of Growth

Order the following sequences from smallest to largest as $n \to \infty$:

$$n!, \ln(\ln(n)), 3n+5, 7, n^{0.0001}, 0.8^n, e^n/200, n^n, \sqrt{n}, \sqrt{9n^2+3n+2}, \ln(n), 1.01^n, n^{100}-2$$

$$0.8^n, 7, \ln\ln n, \ln n, n^{0.0001}, \sqrt{9n^2+3n+2} \sim \sqrt{n}, 3n+5, n^{100}-2, 1.01^n, e^n/200, n!, n^n$$

Determining Convergence of a Sequence

Determine whether each of the following sequences has a limit of 0 or ∞:

1. $\lim_{n\to\infty} e^n/n! = 0$
2. $\lim_{n\to\infty} n^2/\ln(n) = \infty$
3. $\lim_{n\to\infty} \sqrt{n}/1.01^n = 0$
4. $\lim_{n\to\infty} \ln(n)/\ln(\ln(n)) = \infty$
5. $\lim_{n\to\infty} e^n/n^e = \infty$
6. $\lim_{n\to\infty} n^2/\sqrt{n^5+2} = 0$
7. $\lim_{n\to\infty} \frac{n^5 + 10\ln(n)}{5n^3 + 1.01^n} = 0$
8. $\lim_{n\to\infty} \frac{3n + 5 + \ln\ln n}{n!} = 0$
9. $\lim_{n\to\infty} \frac{1.05^n + n^3}{1.04^n + \sqrt{n}} = \infty$
10. $\lim_{n\to\infty} \frac{\sqrt{n^3+1}(n+1)^3}{(n+\ln n)^3} = \infty$
11. $\lim_{n\to\infty} \frac{e^n + e^{-n}}{2n} = \infty$
12. $\lim_{n\to\infty} \frac{n\ln\ln n}{1+\ln n} = \infty$
13. $\lim_{n\to\infty} \frac{e^n \ln n + \sqrt{n}}{n!} = 0$
14. $\lim_{n\to\infty} \frac{1.01^n}{n^{1.01}} = \infty$

Each of the following sequences has a positive finite limit. Find the limit:
(Note: all of these answer come either from mulitplying by the conjugate radical or using L'Hospital's Rule. In the case of $L = \lim_{n\to\infty}(1+1/n)^n$, use $\ln(L) = \lim_{n\to\infty} \ln((1+1/n)^n) = \lim_{n\to\infty} n\ln(1+1/n) = 1$. This is also a good identity to know offhand.)

1. $\lim_{n\to\infty} \sqrt{n^2+2n} - n = 1$

2. $\lim_{n\to\infty} \sqrt{n^2+5n+1} - n = 5/2$

3. $\lim_{n\to\infty} \sqrt{4n^2+6n+3} - 2n = 3/2$

4. $\lim_{n\to\infty} n\ln(1+1/n) = \lim_{x\to 0} \ln(1+x)/x = 1$

5. $\lim_{n\to\infty} n\sin(1/n) = \lim_{x\to 0} \sin(x)/x = 1$

6. $\lim_{n\to\infty}(1+1/n)^n = e$

7. $\lim_{n\to\infty} n(\ln(n+1) - \ln(n)) = \lim_{n\to\infty} n\ln(1+1/n) = 1$

8. $\lim_{n\to\infty} n\sqrt{n^2+1} - n^2 = 1/2$

9. $\lim_{n\to\infty} n(e^{1/n} - 1) = \lim_{x\to 0}(e^x - 1)/x = 1$

True or False?

If true, explain your reasoning. If false, find a counterexample.

1. If $\{a_n\}$ converges, then $\lim_{n\to\infty} a_n = 0$. False: $a_n = 1$
2. If $\{a_n\}$ diverges, then $\lim_{n\to\infty} a_n \neq 0$. True: the limit is $\pm\infty$ or it does not exist.
3. If $\{a_n\}$ converges, then there is an N such that $n \geq N$ implies $|a_n - a_{n+1}| < N$. True: If $\{a_n\}$ converges then $|a_n - a_{n+1}|$ has to become arbitrarily small. If we go by a N-epsilon proof, then we can let $\epsilon = 1/2$ and the result will follow from ther.
4. If $\{a_n\}$ diverges, then for any $\epsilon > 0$ there is an N such that $n \geq N$ implies $|(\lim_{n\to\infty} a_n) - a_n| > \epsilon$. False: The limit $\lim_{n\to\infty} a_n$ does not exist in the first place.
5. If $\lim_{n\to\infty} a_n = 0$ but $\{b_n\}$ diverges, then the sequence $\{a_n b_n\}$ diverges. False: We could have $a_n = 0$ for all n.
6. If $\{a_n\}$ and $\{b_n\}$ both converge, then $\{a_n + b_n\}$ converges. True.
7. If $\{a_n\}$ and $\{b_n\}$ both converge, then $\{a_n b_n\}$ converges. True.
8. If $\{a_n\}$ converges, then $\lim_{n\to\infty} a_n/n = 0$. True.
9. For every $\{a_n\}$ there is some $\{b_n\}$ such that $\{a_n b_n\}$ diverges. False, since $a_n = 0$ serves as a counterexample. If we exclude all sequences that are eventually just a string of zeros, then this statement is true.
10. Every bounded sequence is convergent. False: $a_n = (-1)^n$.
11. Every bounded convergent sequence is monotonic. False: $a_n = (-1)^n/n$
12. If f is a function and $\lim_{n\to\infty} a_n = L$, then $f(L) = \lim_{n\to\infty} f(a_n)$. False: Let $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$

    and let $a_n = 1/n$. Then $\lim_{n\to\infty} a_n = 0$ and $f(0) = 1$, but $f(a_n) = 1$ for all n and so $\lim_{n\to\infty} f(a_n) = 1 \neq f(\lim_{n\to\infty} a_n)$.
13. If f is a continuous function and $\lim_{n\to\infty} a_n = L$, then $f(L) = \lim_{n\to\infty} f(a_n)$. True. This means that the only counterexamples to the previous statement are discontinuous functions.

14. If $\{a_n\}$ converges, then $\{a_n^2\}$ converges.

    True by the previous statement, since $f(x) = x^2$ is a continuous function.

Bonus

1. Prove that $\lim_{n\to\infty} \ln(n)/n^\epsilon = 0$ for any $\epsilon > 0$.

   Use L'Hospital's Rule:

$$\lim_{n\to\infty} \ln(n)/n^\epsilon = \lim_{n\to\infty} \frac{1/n}{\epsilon n^{\epsilon-1}} = \lim_{n\to\infty} \frac{1}{\epsilon n^\epsilon} = 0$$

2. Prove that $\lim_{n\to\infty} k^n/n! = 0$ for any constant $k > 0$.

   Loose proof: the idea is that $k^n/n!$ will keep growing until $n > k$, and will start shrinking afterwards. Let K be an integer such that $K \geq k$. Then

$$\begin{aligned}
\lim_{n\to\infty} k^n/n! &\leq \lim_{n\to\infty} K^n/n! \\
&= \frac{K}{1}\frac{K}{2}\frac{K}{3}\cdots\frac{K}{K}\frac{K}{K+1}\frac{K}{K+2}\cdots \\
\lim_{n\to\infty} k^n/n! &\leq K^K/K! \cdot \lim_{n\to\infty}\left(\frac{K}{K+1}\right)^n \\
&= 0
\end{aligned}$$