11.1/11.2: Sequences/Series Review

Monday, February 23

Speed Round

1. $\lim_{n\to\infty} 2^n/n^2 = \infty$
2. $\lim_{n\to\infty} n/\ln(n) = \infty$
3. $\lim_{n\to\infty} e^n/n! = 0$
4. $\lim_{n\to\infty} n^{100}/n!0$
5. $\lim_{n\to\infty} \ln(n)/\ln(\ln(n)) = \infty$
6. $\lim_{n\to\infty} \ln(n)/n^{0.0001} = 0$
7. $\lim_{n\to\infty} \sin(n)$ DIVERGENT
8. $\lim_{n\to\infty} \sin^2(n) + \cos^2(n) = 1$
9. $\lim_{n\to\infty} \frac{n^2+3n+2}{3n^2+2n+1} = 1/3$
10. $\lim_{n\to\infty} \frac{n+\ln n}{\sqrt{n^2+\ln n}} = 1$
11. $\lim_{n\to\infty} \frac{1.01^n+n^2}{0.95^n+n^5} = \infty$
12. $\lim_{n\to\infty} 0.9999^n = 0$
13. $\lim_{n\to\infty} (-1)^n$ DIVERGENT
14. $\lim_{n\to\infty} \frac{e^n+n}{e^{2n}} = 0$
15. $\lim_{n\to\infty} \frac{n\sqrt{n+1}}{\sqrt{n^3+1}} = 1$
16. $\sum_{n=0}^{\infty} \pi^n$ DIVERGENT
17. $\sum_{n=0}^{\infty} (1/\pi)^n = 1/(1-1/\pi) = \pi/(\pi-1)$
18. $\sum_{n=0}^{\infty} 5/2^n = 5$
19. $\sum_{n=0}^{\infty} (5/2)^n$ DIVERGENT
20. $\sum_{n=0}^{\infty} 1/n$ DIVERGENT

Some Computation Required

1. $\lim_{n\to\infty} \sqrt{n^2-n} - n = -1/2$. (multiply by the conjugate radical)
2. $\lim_{n\to\infty} \sqrt{n^2+1} - n = 0$. (mulitply by the conjugate)
3. $\lim_{n\to\infty} n(\cos(1/n) - 1) = 0$. (l'hops)
4. $\lim_{n\to\infty} n^2(\cos(1/n) - 1) = 1/2$. (l'hops)
5. Prove that if $\epsilon > 0$ then $\lim_{n\to\infty} \ln(n)/n^\epsilon = 0$.

   By L'Hospital's Rule, $\lim_{n\to\infty} \ln(n)/n^\epsilon = \lim_{n\to\infty}(1/n)/\epsilon n^{\epsilon-1} = \lim_{n\to\infty} 1/\epsilon n^\epsilon = 0$.
6. $\sum_{n=1}^{\infty} 3^{n+2}/4^n = 27/4 \sum_{n=1}^{\infty} (3/4)^{n-1} = \frac{27}{4}\frac{1}{1-3/4} = 27.$
7. $\sum_{n=1}^{\infty} 2^{n-2}/5^{n+1} = 1/50 \sum_{n=1}^{\infty} (2/5)^{n-1} \frac{1}{50}\frac{1}{1-2/5} = 1/75.$

Monotone Convergence Theorem

If $\{a_n\}$ is monotonic and bounded, then $\lim_{n\to\infty} a_n$ exists.

1. Draw a picture illustrating the Monotone Convergence Theorem.
2. True or False: $a_n = \ln(n)$ is monotonic, so the MTC implies that $\lim_{n\to\infty} a_n$ exists. False–the sequence is not bounded.
3. True or False: $s_n = \sum_{i=1}^{n} 1/i^2$ is monotonic and bounded above by 2, so the MTC implies that $\lim_{n\to\infty} a_n = 2$. False–the limit exists, but it is not necessarily 2.

4. True of False: $a_n = 1+1/n$ is decreasing and bounded above by 2, so the MTC implies that $\lim_{n\to\infty} a_n$ exists. False–since the sequence is decreasing, we need a lower bound rather than an upper bound for the MTC to apply.

5. If D_n is the world record in the 100-meter dash as of the year n (say, for $n \geq 1900$), what (if anything) does the MTC say about D_n? D_n is monotonic and bounded below by 0, so the MTC says that $\lim_{n\to\infty} D_n$ exists.

6. A North-Going Zax (which only goes north) and a South-Going Zax (which only goes south) are on a collision path. What does the MTC say about the two Zax? Do they necessarily bump into each other? *The position of the North-Going Zax is increasing (if "north" is increasing) and bounded above by the starting position of the South-Going Zax, so the limit of NGZ's position over time exists. Similarly for SGZ. They do not necessarily collide, since they can stop before hitting each other and refuse to move any further.*

Back to Polynomials

1. For what values of x does $\lim_{n\to\infty} x^n = 0$ hold? $|x| < 1$

2. Sketch the graphs of the functions $f(x) = x^{100}, g(x) = x^{101}, h(x) = 0$. When are the first two "good" approximations of the third?

3. For what values of x does $\sum_{n=0}^{\infty} x^n = 1/(1-x)$ hold? $|x| < 1$

4. Sketch the graphs of the functions $f(x) = 1+x+x^2+\ldots x^{50}, g(x) = 1+x+x^2+\ldots+x^{51}, h(x) = 1/(1-x)$. When are the first two "good" approximations of the third?

5. For what values of x does $\lim_{n\to\infty}(x/2)^n = 0$ hold? $|x| < 2$

6. For what values of x does $\sum_{n=0}^{\infty}(x/2)^n = 1/(1-x/2)$ hold? $|x| < 2$

7. What is $\sum_{n=0}^{\infty} x^{2n}$, when $|x| < 1$? $1 + x^2 + x^4 + \ldots$