9.3: Separable Equations

Friday, April 10

Review

1. Show that $y = Ae^{2x} + Be^{-2x}$ satisfies the differential equation $y'' = 4y$ for any constants A, B.

$$y'' = (Ae^{2x} + Be^{-2x})'' = 4Ae^{2x} + 4Be^{2x} = 4(Ae^{2x} + Be^{-2x}) = 4y$$

2. Show that $y = \ln x$ satisfies the differential equation $y' = e^{-y}$.

   $y' = 1/x = 1/e^{\ln x} = e^{-\ln x} = e^{-y}$.

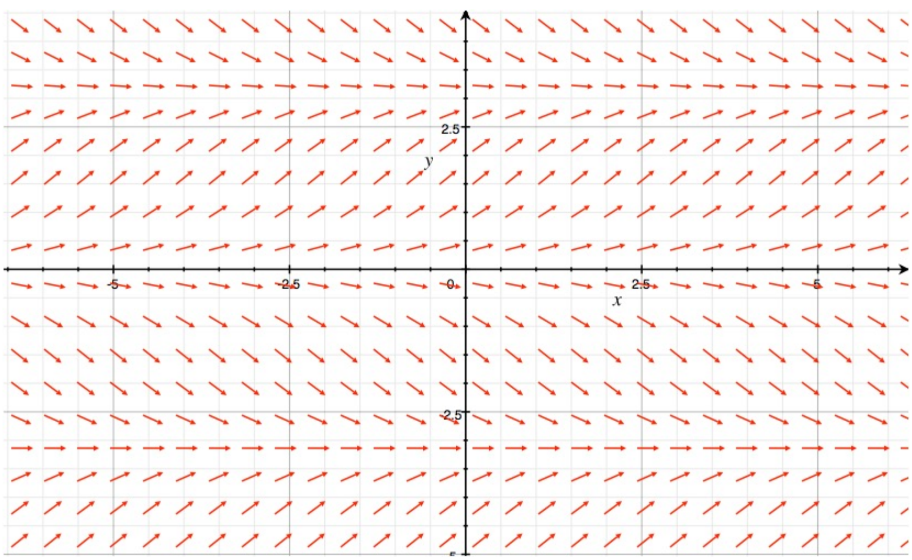
3. Suppose a particle's velocity is described by $\frac{dx}{dt} = \sin x$. If $x(0) = 1$, what will happen to the particle as $t \to \infty$?

   The particle will approach $x = \pi$.

4. What if $x(0) = 2\pi$? What if $x(0) = -1$?

   If $x(0) = 2\pi$, the particle will stay still. If $x(0) = -1$, the particle will approach $x = -\pi$.

5. Sketch a direction field for the $\frac{dx}{dt} = \sin x$, including lines where $\frac{dx}{dt} = 0$.



Separable Equations

Find the solution of the differential equation that satisfies the given initial condition.

1. $\frac{dy}{dx} = y, y(0) = 2$

$$\begin{aligned} \frac{1}{y}\,dy &= dx \\ \ln y &= x + C \\ y &= e^{x+C} \\ &= ke^{x} \\ &= 2e^{x} \end{aligned}$$

The final equality comes from solving $y(0) = 2$, the initial condition.

2. $\frac{dy}{dx} = xy, y(0) = 1$

$$\begin{aligned} \frac{1}{y}\,dy &= x\,dx \\ \ln y &= \frac{1}{2}x^2 + c \\ y &= e^{x^2/2+c} \\ &= ke^{x^2/2} \\ &= e^{x^2/2} \end{aligned}$$

The final equality (with $k = 1$) comes form solving $y(0) = 1$.

3. $\frac{dy}{dx} = (1+y)/x, y(1) = 3$

$$\begin{aligned} \frac{1}{1+y}\,dy &= \frac{1}{x}\,dx \\ \ln(1+y) &= \ln x + c \\ 1 + y &= e^{\ln x + c} \\ y &= -1 + kx \\ y &= 4x - 1 \end{aligned}$$

(Note: The problem originally had the initial condition $y(0) = 1$, which would have no solutions)

4. $\frac{dy}{dx} = 1 + x + y + xy, y(0) = -1$

$$\begin{aligned} \frac{dy}{dx} &= (1+x)(1+y) \\ \frac{1}{1+y}\,dy &= (1+x)\,dx \\ \ln(1+y) &= c + x + x^2/2 \\ y &= -1 + e^{c+x+x^2/2} \end{aligned}$$

This might appear to have no solutions given $y(0) = -1$, but as it turns out the constant $y = -1$ is itself a solution.

5. $\frac{dy}{dx} = e^{y+x}, y(0) = 2$

$$\begin{aligned} e^{-y}\,dy &= e^x\,dx \\ -e^{-y} &= c + e^x \\ e^{-y} &= -c - e^x \\ y &= -\ln(-c - e^x) \\ y &= -\ln(1 - 1/e - e^x) \end{aligned}$$

   Note that this solution will not be defined for all values of x, since $\ln(x)$ is only defined for positive numbers.

6. $\frac{dy}{dx} = y + x, y(0) = 0$ (Hint: Substitute $u = y + x$)

$$\begin{aligned} u &= y + x \\ \frac{du}{dx} &= \frac{dy}{dx} + 1 \\ \frac{du}{dx} &= 1 + u \\ \frac{1}{1+u}\,du &= dx \\ \ln(1 + u) &= x + c \\ 1 + u &= ke^x \\ y &= ke^x - x - 1 \end{aligned}$$

Back to Air Resistance

Say we throw a pillow off a cliff. Acceleration due to gravity is constant, but air resistance is greater the faster the pillow moves. For this reason, we'll model the acceleration on the pillow as

$$a = \frac{dv}{dt} = -10 - v.$$

1. Find an equation for $v(t)$, assuming the initial velocity of the pillow is zero.

$$\begin{aligned} \frac{1}{10+v}\,dv &= -dt \\ \ln(10 + v) &= c - t \\ v &= -10 + ke^{-t} \end{aligned}$$

2. Find an equation for $y(t)$, the height of the pillow over time.

$$
\begin{aligned}
\frac{dy}{dt} &= -10 + ke^{-t} \\
dy &= (-10 + ke^{-t})\, dt \\
\int_{t=0}^{T} dy &= \int_{t=0}^{T} (-10 + ke^{-t})\, dt \\
y - y_0 &= -10t + k(1 - e^{-t}) \\
y &= y_0 - 10t + k(1 - e^{-t})
\end{aligned}
$$

3. What happens to $v(t)$ and $y(t)$ as $t \to \infty$? Assume the cliff overlooks an infinite abyss.

   As $t \to \infty$, $v \to -10$ and $y \to -\infty$... the object will eventually fall at a more or less constant rate.

Melting Snowball

A spherical snowball sits in the hot sun. Since only the surface of the snowball is exposed to the sunlight, assume that the snowball melts at a rate proportional to its surface area. After 1 hour, the snowball is half its original size.

1. Set up a differential equation in terms of volume, surface area, and time.

$$\frac{dV}{dt} = kA$$

2. Put the previous equation in terms of time and the radius of the snowball.

$$
\begin{aligned}
\frac{dV}{dr}\frac{dr}{dt} &= k \cdot 4\pi r^2 \\
d(4/3\pi r^3)/dr \frac{dr}{dt} &= k \cdot 4\pi r^2 \\
4\pi r^2 \frac{dr}{dt} &= k \cdot 4\pi r^2 \\
\frac{dr}{dt} &= k
\end{aligned}
$$

   The snowball's radius decreases at a constant rate!

3. How long does the snowball take to melt entirely?

   Since $V \sim r^3$, if the volume has been cut in half then that means the radius after one hour is $1/\sqrt[3]{2}$ the original radius. If the change in radius is constant over time, then the snowball should take $1/(1 - \sqrt[3]{2}) \approx 4.85$ hours to melt.

Spreading Rumor

Model the rate at which a rumor spreads with $\frac{dP}{dt} = P(1 - P)$, where P is the perecentage of people who have heard the rumor.

$$
\begin{aligned}
\frac{1}{P(1-P)}\,dP &= dt \\
\left(\frac{1}{1-P} + \frac{1}{P}\right) dP &= dt \\
-\ln(1-P) + \ln P &= t + c \\
\ln(P/(1-P)) &= t + c \\
P/(1-P) &= e^{t+c} \\
1/(1-P) - 1 &= ke^t \\
1/(1-P) &= 1 + ke^t \\
1 - P &= \frac{1}{1+ke^t} \\
P &= 1 - \frac{1}{1+ke^t}
\end{aligned}
$$

1. Solve for $P(t)$ where $P(0) = 1/10$.

$$
\begin{aligned}
1/10 &= 1 - \frac{1}{1+ke^t} \\
\frac{1}{1+ke^t} &= 9/10 \\
1 + ke^0 &= 10/9 \\
k &= 1/9 \\
P &= 1 - \frac{1}{1+e^t/9}
\end{aligned}
$$

2. Check that this equation qualitatively behaves in the manner you would expect (how do you expect it to behave?)

   (a) The term $(1 + e^t/9)$ is increasing over time, so $\frac{1}{1+e^t/9}$ is decreasing and therefore $P = 1 - \frac{1}{1+e^t/9}$ is strictly increasing over time.

   (b) As $t \to \infty$, P approaches 1.

   (c) P $\leq$ 1 for all time.