11.8: Power Series!

Wednesday, March 11

Speed Round

Determine whether each of the following series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{1}{n}$ DIV
2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ CC
3. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ ABS
4. $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ DIV
5. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ DIV
6. $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\ln n}$ CC
7. $\sum_{n=1}^{\infty} \frac{n^2 \ln n}{4^n}$ ABS
8. $\sum_{n=1}^{\infty} \frac{n^5 5^n}{n!}$ ABS
9. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^3}$ DIV
10. $\sum_{n=1}^{\infty} (-1)^n$ DIV
11. $\sum_{n=1}^{\infty} \frac{3^n}{n^2+\sqrt{n}}$ DIV
12. $\sum_{n=1}^{\infty} \frac{n!}{30^n}$ DIV

Interval of Convergence

How to tell at a glance what the interval of convergence is:

1. If $n!$ appears, ignore everything else. $R = 0$ or $R = \infty$.
2. If not, write your series (if possible) as $\sum_{n=1}^{\infty} A(n) \cdot (x-a)^n \cdot r^n$ where r^n is the part that increases exponentially and $A(n)$ is everything else.
3. $R = 1/|r|$. The interval of convergence is $(a-R, a+R)$, except maybe for the endpoints.
4. If $\sum_{n=1}^{\infty} A(n)$ converges absolutely, the interval is $[a-R, a+R]$.
5. If $\sum_{n=1}^{\infty} A(n)$ converges conditionally the interval will be one-sided (either $(a-R, a+R]$ or $[a-R, a+R)$).
6. If $\lim_{n\to\infty} A(n) \neq 0$, the interval is $(a-R, a+R)$.

Find the intervals of convergence of the following functions:

1. $\sum_{n=1}^{\infty} x^n$: $(-1, 1)$
2. $\sum_{n=1}^{\infty} \frac{x^n}{n}$: $[-1, 1)$
3. $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$: $[-1, 1]$
4. $\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n}$: $(1, 5)$
5. $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n \cdot 3^n}$: $[2, 12)$
6. $\sum_{n=1}^{\infty} (-2)^n \frac{(x+4)^n}{\sqrt{n}}$: $(-4.5, -3.5]$
7. $\sum_{n=1}^{\infty} 5^n \frac{(x+3)^n}{n!}$: $(-\infty, \infty)$
8. $\sum_{n=1}^{\infty} (x+7)^n \frac{n}{2^n}$: $(-9, -5)$
9. $\sum_{n=1}^{\infty} n!(x-1)^n$: $\{1\}$ only.
10. $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n(-4)^n}$: $(-1, 7]$
11. $\sum_{n=1}^{\infty} \frac{x^n}{n!}$: $(-\infty, \infty)$
12. $\sum_{n=1}^{\infty} (-1)^n \frac{(x+2)^n}{n\sqrt{n}}$: $[-3, -1]$

Power Series Arithmetic!

1. $e^x = 1 + x + x^2/2! + x^3/3! - \ldots = \sum_{n=0}^{\infty} x^n/n!$
2. $\sin x = x - x^3/3! + x^5/5! - \ldots = \sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)!$
3. $\cos x = 1 - x^2/2! + x^4/4! - \ldots = \sum_{n=0}^{\infty}(-1)^n x^{2n}/(2n)!$
4. $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \displaystyle\sum_{n=0}^{\infty} x^n$

Rewrite the power series above in sigma notation.

Add and Multiply!

1. $5e^x = 5 + 5x + 5x^2/2! + 5x^3/3! - \ldots$
2. $\sin x + \cos x = 1 + x - x^2/2! - x^3/3! + x^4/4! + x^5/5! - \ldots$

Compose!

1. $e^{-x^2} = 1 - x^2 + x^4/2! - x^6/3! + \ldots$
2. $\sin(2x) = 2x - (2x)^3/3! + (2x)^5/5! - \ldots$
3. $\dfrac{1}{1-3x} = 1 + (3x) + (3x)^2 + (3x)^3 + \ldots = \displaystyle\sum_{n=0}^{\infty}(3x)^n$
4. $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$
5. $\dfrac{1}{5-x} = \dfrac{1}{5}(1 + (x/5) + (x/5)^2 + (x/5)^3 + \ldots)$
6. $\dfrac{1}{2+3x} = \dfrac{1}{2}(1 - (3x/2) + (3x/2)^2 - (3x/2)^3 + \ldots)$

Differentiate and Integrate!

1. $\frac{d}{dx}e^x = e^x$
2. $\frac{d}{dx}\sin x = \cos x$
3. $\int e^x = e^x$
4. $\ln(1+x) = \int \frac{1}{1+x} = 1 - x + x^2/2 - x^3/3 + \ldots$
5. $\arctan x = \int \frac{1}{1+x^2} = 1 - x^3/3 + x^5/5 - x^7/7 + \ldots$
6. $\int \sin x = -\cos x$

. . . Multiply and Divide?

Verify the following:

1. $\sin(2x) = 2\sin x \cos x$
2. $e^x e^{-x} = 1$
3. $\sin^2 x + \cos^2 x = 1$
4. Find $1/\cos(x)$ by solving $(a_0 + a_1 x + a_2 x^2 + \ldots)\cos(x) = 1$ term-by-term.

Invert??

1. Find $\arcsin(x)$ by solving $P(\sin x)$ term-by-term.
2. Find $\sqrt{1+x}$ by solving $P(x)^2 = 1 + x$.