Math 1B, Quiz 7

Monday, March 16

1. (1 pt each) Give power series representations for the following functions (centered at $x = 0$) up and including the x^3 term.

   (a) $e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots$

   (b) $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$

2. (3 pts) A power series centered at $x = 5$ converges at $x = 2$ and diverges at $x = 10$. For each of the following points state whether the series converges or diverges at that point, or whether there is not enough information to tell.

   (a) $x = -1$ diverges

   (b) $x = 1$ unknown

   (c) $x = 7$ converges

   (d) $x = 8$ unknown

   (e) $x = 9$ unknown

   (f) $x = 11$ diverges

3. (3 pts) Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2 \cdot 3^n}$. Show your work.

$$\begin{aligned} \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \frac{|x-2|}{3} \\ &= \frac{|x-2|}{3} \end{aligned}$$

   So the series converges when $|x-2|/3 < 1$, meaning $|x-2| < 3$ and so $-1 < x < 5$. Checking the endpoints gives the two series $\sum_{n=1}^{\infty} \frac{(\pm 1)^n}{n^2}$, both of which converge absolutely. The interval of convergence is therefore $[-1, 5]$.

   Shortcut: The interval of convergence has center $x = 2$ and radius 3, and converges at both endpoints because $\sum_{n=1}^{\infty} 1/n^2$ converges absolutely.

4. (3 pts) Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(3-2x)^n}{n}$. Show your work.

$$\begin{aligned} \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{n+1}{n} |3-2x| \\ &= |3-2x| \end{aligned}$$

   So the series converges when

$$\begin{aligned} |3-2x| &< 1 \\ -1 < 3-2x &< 1 \\ -4 < -2x &< -2 \\ 1 < x &< 2 \end{aligned}$$

When $x = 2$ the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ and so converges. When $x = 1$ the series becomes $\sum_{n=1}^{\infty} \frac{1}{n}$ and so diverges. The interval of convergence is therefore $(1, 2]$.

Note: Re-writing the series as $\frac{1}{n} 2^n (\frac{3}{2} - x)^n$ allows for the shortcut observation that the interval has center $3/2$ and radius $1/2$. Since the non-exponential part is $1/n$, it should converge on one edge only.

Extra Credit

Give as many terms as you can of the Taylor series for $\cos^2(x) + \sin^2(x)$ centered at $x = 0$ (0.1 pt per term, max. 0.5 pts).
Answer: 1