Math 1B, Quiz 3–Solutions

Monday, February 9

Name:

1. (1 pt) Evaluate the integral $\int \frac{3x+2}{1+x^2}\,dx$.

   Answer: $\frac{3}{2}\ln(1+x^2) + 2\arctan(x)$.

2. (3 pts) Evaluate the integral $\int xe^x\,dx$.

   Pick $u = x, dv = e^x dx$. Then $\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x$.

3. (3 pts) Evaluate the integral $\int \frac{x}{\sqrt{1+x^2}}\,dx$.

   Method 1: Substitute $u = 1+x^2, du = 2x\,dx$. Then $\int \frac{x}{\sqrt{1+x^2}}\,dx = \int \frac{1}{2\sqrt{u}}\,du = \sqrt{u} = \sqrt{1+x^2}$.

   Method 2: Substitute $x = \tan\theta, dx = \sec^2\theta$. Then

$$\begin{aligned}\int \frac{x}{\sqrt{1+x^2}}\,dx &= \int \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}\sec^2\theta\,d\theta \\ &= \int \tan\theta\sec\theta\,d\theta \\ &= \sec\theta \\ &= \sqrt{1+x^2}\end{aligned}$$

4. (3 pts) Evaluate the integral $\int \frac{x^2+3x+5}{(x+1)^2}\,dx$.

   First we have to use long division to make the function a proper rational function. Doing this gives $\int \frac{x^2+3x+5}{(x+1)^2}\,dx = \int 1 + \frac{x+4}{(x+1)^2}\,dx$.
   Using the method of partial fractions, we get

$$\begin{aligned}\frac{x+4}{(x+1)^2} &= \frac{A}{x+1} + \frac{B}{(x+1)^2} \\ x+4 &= A(x+1) + B\end{aligned}$$

   Solving gives $A = 1, B = 3$ (Alternatively, since $(x+1)^2$ is the only factor in the denominator we could have gotten this by dividing $(x+4)$ by $(x+1)$). Thus our final answer is

$$\int \frac{x^2+3x+5}{(x+1)^2}\,dx = \int 1 + \frac{1}{x+1} + \frac{3}{(x+1)^2} = x + \ln(x+1) - \frac{3}{x+1}$$

5. (3 pts) How many intervals must we use (how large must n be) to guarantee that the Midpoint rule approximation to $\int_0^2 e^{x^2}\,dx$ is accurate to within 0.001 (1/1000)?

The derivative of e^{x^2} is $2xe^{x^2}$, and the second derivative is $2e^{x^2} + 4x^2e^{x^2} = (4x^2+2)e^{x^2}$, which is bounded on the interval $[0,2]$ by $18e^4$, which is less than 1800 since $e < 3$ and so $e^4 < 100$. Thus we chose $K = 1800$ (if we wanted a tighter bound, using a calculator to find e^4 shows that $K = 1000$ would also work).

So $K = 1800$, $(b-a) = 2$, and the error bound for the midpoint rule is $|E_M| \leq \frac{K(b-a)^3}{24n^2}$. Since we want the error to be less than 0.001, we want

$$\begin{aligned} 0.001 &\geq \frac{1800 \cdot 8}{24n^2} \\ n^2 &\geq 600,000 \\ n &\geq 800 \end{aligned}$$

So $n = 800$ would be sufficient to guarantee an error of at most 0.001.

6. (2 pts) Identify all of the following integrals as convergent or divergent:

   (a) $\int_1^\infty \frac{1}{x}\,dx$: DIVERGENT

   (b) $\int_1^\infty \frac{1}{x^2}\,dx$: CONVERGENT

   (c) $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$: DIVERGENT

   (d) $\int_0^1 \frac{1}{x}\,dx$: DIVERGENT

   (e) $\int_0^1 \frac{1}{x^2}\,dx$: DIVERGENT

   (f) $\int_0^1 \frac{1}{\sqrt{x}}\,dx$: CONVERGENT

7. (4 pts) Mark the following statements as true or false. You do not need to show your work.

   (a) $\int_1^\infty \frac{\sin^2 x}{x^3}\,dx$ converges by comparison with $\int_1^\infty \frac{1}{x^3}\,dx$. TRUE

   (b) $\int_1^\infty \frac{\sin x}{x}\,dx$ diverges by comparison with $\int_1^\infty \frac{1}{x}\,dx$. FALSE

   (c) $\int_0^1 \frac{\ln(1+x)}{x}\,dx$ diverges by comparison with $\int_0^1 \frac{1}{x}\,dx$. FALSE

   (d) $\int_0^\infty \frac{1}{(x-1)^2}\,dx$ is a divergent improper integral. TRUE

Extra Credit

Mark all statements as true or false (0.1 pt each). Answers will be judged based on their consistency with your other answers rather than according to a theoretical "correct" solution.

1. At least three of these statements are true. **False**
2. At least three of these statements are false. **True**
3. Statements 1 and 2 have the same answer. **False**
4. This statement and statement 5 have different answers. **True**
5. Exactly one of these statements is true. **False**