Practice Final: Solutions

Wednesday, May 6

1. Evaluate the integral $\int \cos x \ln \sin x \, dx$.

   First substitute $y = \sin x$, then use integration by parts with $u = \ln y, dv = dy$.

$$\begin{aligned}
\int \cos x \ln \sin x \, dx &= \int \ln y \, dy \\
&= y \ln y - \int 1 \, dy \\
&= y \ln y - y \\
&= \sin x \ln \sin x - \sin x
\end{aligned}$$

2. Evaluate the integral $\int \frac{2}{(x^2+1)(x+1)}$.

$$\begin{aligned}
\frac{2}{(x^2+1)(x+1)} &= \frac{Ax+B}{x^2+1} + \frac{C}{x+1} \\
2 &= (Ax+B)(x+1) + C(x^2+1) \\
2 &= 2C \\
C &= 1 \\
1 &= Ax^2 + (A+B)x + B + x^2 + 1 \\
A &= -1 \\
B &= 1 \\
\int \frac{2}{(x^2+1)(x+1)} &= \int \frac{-x+1}{x^2+1} + \frac{1}{x+1} \\
&= \int -\frac{x}{x^2+1} + \frac{1}{x^2+1} + \frac{1}{x+1} \\
&= -\frac{1}{2}\ln(x^2+1) + \arctan(x) + \ln(x+1)
\end{aligned}$$

3. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{2n+1}(3x-1)^n$.

$$\begin{aligned}
\lim_{n\to\infty} |a_{n+1}/a_n| &= \lim_{n\to\infty} \frac{(2n+1)\sqrt{n+2}}{(2n+3)\sqrt{n+1}}|3x-1| \\
&= \lim_{n\to\infty} |3x-1|
\end{aligned}$$

   So for the series to converge by the Ratio Test, we need to have

$$\begin{aligned}
|3x-1| &< 1 \\
|x-1/3| &< 1/3 \\
0 < x &< 2/3
\end{aligned}$$

   When $x = 0$, the series $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{2n+1}(-1)^n$ converges by the Alternating Series test.

When $x = 2/3$ the series $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{2n+1}$ diverges by Limit Comparison Test with $b_n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$.

The interval of convergence is therefore $[0, 2/3)$.

4. Solve the initial-value problem $e^x y' = 1 + y, y(0) = -1$. [UPDATED: Original posted solution was incorrect]

$$\begin{aligned} e^x \frac{dy}{dx} &= 1 + y \\ \frac{dy}{1+y} &= e^{-x} dx \\ \int \frac{dy}{1+y} &= \int e^{-x}\, dx \\ \ln(|1+y|) &= -e^{-x} + C \\ y &= Ke^{-e^{-x}} - 1 \\ y &= -1 \end{aligned}$$

5. Find the general solution to the equation $y'' + y = \frac{1}{\sin x}$

$$\begin{aligned} y_1, y_2 &= \cos x, \sin x \\ W &= 1 \\ y &= -\cos x \int dx + \sin x \int \frac{\cos x}{\sin x}\, dx \\ &= -\cos x(x + C) + \sin x(\ln \sin x + C') \\ &= \sin x \ln \sin x - x \cos x + C_1 \cos x + C_2 \sin x \end{aligned}$$

6. True or False: you do not have to show your work if the answer is true, but give a counterexample if the answer is false.
   (a) If $\{a_n\}$ is any sequence and $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n b_n = 0$.
   False: $a_n = n, b_n = 1/n$.
   (b) If $\lim_{n\to\infty} |a_{n+1}|/|a_n| = 1$ then $\sum_{n=1}^{\infty} a_n$ converges conditionally.
   False: $a_n = 1/n^2$.
   (c) If $a_n, b_n > 0$ and $\lim_{n\to\infty} a_n/b_n = 0$ and $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.
   False: $a_n = 1/n^2, b_n = 1$.
   (d) If $a_n > a_{n+1} > 0$ for all n then $\lim_{n\to\infty} a_n$ exists.
   True, by the Monotone Convergence Theorem.

7. Mark each integral or series as convergent or divergent. You do not have to show your work.
   (a) $\int_0^{\infty} \frac{1}{\sqrt{|x-3|}}\, dx$: DIV
   (b) $\int_1^{\infty} \frac{1}{x^2}\, dx$: CONV
   (c) $\int_1^{\infty} \frac{\sqrt{x} + e^{-x}}{x + \ln x}\, dx$: DIV
   (d) $\int_0^{1} \frac{1}{\sin x}\, dx$: DIV
   (e) $\sum_{n=1}^{\infty} \frac{n^3 + 3^n}{n!}$: CONV
   (f) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$: CONV