Partial Fractions

Friday, January 30

Long Division

Simplify the following expressions:

1. $\dfrac{x^3 + 2x^2 + x + 1}{x^2}$
2. $\dfrac{x^3 + 1}{x + 1}$
3. $\dfrac{x^3 - 1}{x - 2}$
4. $\dfrac{x^2 + x + 1}{x(x - 2)}$
5. $\dfrac{2x^2 - 1}{x^2 + 1}$
6. $\dfrac{x^3 + x^2 + x}{x(x^2 + 1)}$

Factoring

1. Rational Roots Theorem: If $F(x) = Ax^n + c_{n-1}x^{n-1} + \ldots + c_1 x + B = 0$ is a polynomial, then for each rational solution $x = p/q$ (with p/q in lowest terms), A will be divisible by q and B will be divisible by p (note that if $F(p/q) = 0$ then $(qx - p)$ is a factor of $F(x)$).
2. For any given quadratic $ax^2 + bx + c$, if $b^2 - 4ac < 0$ the quadratic is irreducible (so look for an arctan-type expression). Otherwise, the polynomial factors as $(x - \alpha)(x - \beta)$, where $\alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Factor the following quadratics if they are reducible, and put them in the form $(ax + b)^2 + c$ if they are not.

1. $x^2 + 2x + 5$
2. $3x^2 - 5x - 2$
3. $x^2 + x - 2$
4. $x^2 - 2x + 2$
5. $2x^2 + x - 3$
6. $4x^2 - 4x + 2$
7. $x^2 + 6x + 9$
8. $6x^2 + x - 2$

Putting in Partial Fraction Form

Be careful if there are repeated roots! Be careful with irreducible quadratics! As a template:

$$\frac{P}{x^3(x^2 + 1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx + E}{x^2 + 1} + \frac{Fx + G}{(x^2 + 1)^2}$$

Two methods for decomposing a rational function into partial fractions:

$$\frac{3x + 2}{(x - 1)(x + 2)} = \frac{A}{x - 1} + \frac{B}{x + 2}$$
$$3x + 2 = A(x + 2) + B(x - 1)$$

1. Get the system of equations $(A + B)x = 3x, (2A - B) = 2$, and solve for A and B.
2. Plug in strategic values of x (here, $x = 1$ and $x = -2$)

Try decomposing $\frac{2x+1}{(x+1)(x-3)}$ using both methods.

Partial Fraction Battle!

Which way works better? Let's find out!

1. $x+1 = A(x+2) + B(x+3)$

2. $x-4 = Ax + B(x+7)$

3. $x^2+2x-1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$

4. $x^2+3x = A(x+4)(x-1) + B(x-1)(2x+1) + C(x+4)(2x+1)$

5. $x^2+x+1 = Ax(x+3) + B(x+3) + Cx^2$

6. $x+1 = A(x^2+4) + (Bx+C)x$

Integrating

To integrate $\dfrac{Px+Q}{(ax+b)^2+c}$:

1. Substitute $u = (ax+b)$ to get something of the form $\dfrac{Ru+S}{u^2+c}$
2. Split into $\dfrac{Ru}{u^2+c} + \dfrac{S}{u^2+c}$
3. Substitute $v = 2u$ for the first, $u = \tan(t)\sqrt{c}$ for the second.

1. $\displaystyle\int \frac{x^2+2x+3}{x^2+1}$
2. $\displaystyle\int \frac{x^2+2x+3}{x^2}$
3. $\displaystyle\int \frac{1}{(x-3)(x+1)}$
4. $\displaystyle\int \frac{x+1}{x(x^2+1)}$
5. $\displaystyle\int \frac{x^2+5}{(x+1)(x^2+2x+2)}$
6. $\displaystyle\int \frac{-x^3}{x(x+1)^2}$