Partial Fractions

Friday, January 30

Long Division

Simplify the following expressions:

1. $\dfrac{x^3+2x^2+x+1}{x^2}$

   Answer: $x+2+\dfrac{1}{x}+\dfrac{1}{x^2}$

2. $\dfrac{x^3+1}{x+1}$

   Answer: x^2-x+1

3. $\dfrac{x^3-1}{x-2}$

   Answer: $x^2+2x+4+\dfrac{7}{x-2}$

4. $\dfrac{x^2+x+1}{x(x-2)}$

   Answer: $1+\dfrac{3x+1}{x^2-2x}$

5. $\dfrac{2x^2-1}{x^2+1}$

   Answer: $2-\dfrac{3}{x^2+1}$

6. $\dfrac{x^3+x^2+x}{x(x^2+1)}$

   Answer: $1+\dfrac{x^2}{x^3+x}$

Factoring

1. Rational Roots Theorem: If $F(x) = Ax^n + c_{n-1}x^{n-1} + \ldots + c_1 x + B = 0$ is a polynomial, then for each rational solution $x = p/q$ (with p/q in lowest terms), A will be divisible by q and B will be divisible by p (note that if $F(p/q) = 0$ then $(qx - p)$ is a factor of $F(x)$).
2. For any given quadratic ax^2+bx+c, if $b^2-4ac<0$ the quadratic is irreducible (so look for an arctan-type expression). Otherwise, the polynomial factors as $(x-\alpha)(x-\beta)$, where $\alpha,\beta = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Factor the following quadratics if they are reducible, and put them in the form $(ax+b)^2+c$ if they are not.

1. x^2+2x+5

   Not reducible: $(x+1)^2+4$

2. $3x^2-5x-2$

   Reducible: $(3x+1)(x-2)$

3. x^2+x-2

   Reducible: $(x+2)(x-1)$

4. x^2-2x+2

   Not reducible: $(x-1)^2+1$

5. $2x^2+x-3$

   Reducible: $(2x+3)(x-1)$

6. $4x^2-4x+2$

   Not reducible: $(2x-1)^2+1$

7. x^2+6x+9

   Reducible: $(x+3)^2$

8. $6x^2+x-2$

   Reducible: $(3x+2)(2x-1)$

Putting in Partial Fraction Form

Be careful if there are repeated roots! Be careful with irreducible quadratics! As a template:

$$\frac{P}{x^3(x^2+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2}$$

Two methods for decomposing a rational function into partial fractions:

$$\frac{3x+2}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$
$$3x+2 = A(x+2) + B(x-1)$$

1. Get the system of equations $(A+B)x = 3x, (2A-B) = 2$, and solve for A and B.
2. Plug in strategic values of x (here, $x = 1$ and $x = -2$)

Try decomposing $\frac{2x+1}{(x+1)(x-3)}$ using both methods.

$$\text{Answer: } \frac{2x+1}{(x+1)(x-3)} = \frac{1}{4}\left(\frac{1}{x+1}\right) + \frac{7}{4}\left(\frac{1}{x-3}\right)$$

Partial Fraction Battle!

Which way works better? Let's find out!

1. $x+1 = A(x+2) + B(x+3)$
   Answer: $A = 2, B = -1$
2. $x-4 = Ax + B(x+7)$
   Answer: $A = 11/7, B = -4/7$
3. $x^2 + 2x - 1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$
   Answer: $A = 1/2, B = 1/10, C = -1/10$
4. $x^2 + 3x = A(x+4)(x-1) + B(x-1)(2x+1) + C(x+4)(2x+1)$
   Answer: $A = 5/21, B = 4/35, C = 4/15$
5. $x^2 + x + 1 = Ax(x+3) + B(x+3) + Cx^2$
   Answer: $A = 2/9, B = 1/3, C = 7/9$
6. $x+1 = A(x^2+4) + (Bx+C)x$
   Answer: $A = 1/4, B = -1/4, C = 1$

Integrating

To integrate $\frac{Px+Q}{(ax+b)^2+c}$:

1. Substitute $u = (ax+b)$ to get something of the form $\frac{Ru+S}{u^2+c}$
2. Split into $\frac{Ru}{u^2+c} + \frac{S}{u^2+c}$
3. Substitute $v = 2u$ for the first, $u = \tan(t)\sqrt{c}$ for the second.

1. $\int \frac{x^2+2x+3}{x^2+1}$

   Answer: $x + \ln(x^2+1) + 2\arctan(x)$

2. $\int \frac{x^2+2x+3}{x^2}$

   Answer: $x + 2\ln(x) - 3/x$

3. $\int \frac{1}{(x-3)(x+1)}$

Answer: $\frac{1}{4}\ln(x-3) - \frac{1}{4}\ln(x+1)$

4. $\int \frac{x+1}{x(x^2+1)}$

Answer: $\ln(x) - \frac{1}{2}\ln(1+x^2) + \arctan(x)$

5. $\int \frac{x^2+5}{(x+1)(x^2+2x+2)}$

Answer: $6\ln(x+1) - \frac{5}{2}\log(x^2+2x+2) - 2\arctan(x+1)$

6. $\int \frac{-x^3}{x(x+1)^2}$

Answer: $2\ln(x+1) - x + \frac{1}{x+1}$