Numerical Integration

Wednesday, February 4

Trig

1. $\sin^2\theta + \cos^2\theta = 1$
2. $\tan^2\theta + 1 = \sec^2\theta$
3. $\sec^2\theta - 1 = \tan^2\theta$
4. $\sec\theta\cos\theta = 1$
5. $\tan\theta\cos\theta = \sin\theta$
6. $\tan\theta/\sec\theta = \sin\theta$
7. $x = \sin\theta; \tan\theta = \sqrt{x^2 - 1}$
8. $x = \tan\theta; \sec\theta = \sqrt{x^2 + 1}$
9. $x = \tan\theta; \sin\theta = \frac{x}{\sqrt{1+x^2}}$
10. $\int 1/\sec\theta\, d\theta = \sin\theta$
11. $\int 1/\cos^2\theta\, d\theta = \tan\theta$
12. $\int \tan\theta\cos\theta\, d\theta = -\cos\theta$

arctan(x), ln(x)

1. $\int \frac{3x+5}{x^2+1} = \frac{3}{2}\ln(x^2+1) + 5\arctan(x)$
2. $\int \frac{4x-1}{x^2+1} = 2\ln(x^2+1) - \arctan(x)$
3. $\int \frac{7x+2}{x^2+1} = \frac{7}{2}\ln(x^2+1) + 2\arctan(x)$
4. $\int \frac{2x-4}{x^2+1} = \ln(x^2+1) + -4\arctan(x)$
5. $\int \frac{x^2+2x+2}{x^2+1} = x + \ln(x^2+1) + \arctan(x)$
6. $\int \frac{x^2+3x-5}{x^2+1} = x + \frac{3}{2}\ln(x^2+1) - 6\arctan(x)$

Repeated Roots

1. $\int \frac{x^2+3x+2}{x^2} = \int 1 + 3/x + 2/x^2 = x + 3\ln(x) - 2/x$
2. $\int \frac{x^2-x+4}{x^2} = \int 1 - 1/x + 4/x^2 = x - \ln(x) - 4/x$
3. $\int \frac{x^2+5}{(x-1)^2} = \int 1 + 2/(x-1) + 6/(x-1)^2 = x + 2\ln(x-1) - 6/(x-1)$
4. $\int \left(\frac{x}{x-2}\right)^2 = \int 1 + 4/(x-2) + 4/(x-2)^2 = x + 4\ln(x-2) - 4/(x-2)$
5. $\int \frac{3x^2+4x+7}{(x+1)^2} = \int 3 - 2/(x+1) + 6/(x+1)^2 = 3x - 2\ln(x+1) - 6/(x+1)$
6. $\int \frac{5x^2+5x+1}{(x+1)^2} = \int 5 - 5/(x+1) + 1/(x+1)^2 = 5x - 5\ln(x+1) - 1/(x+1)$

Numerical Integration

The template for pretty much every problem, according to practice midterms:

"How many intervals do you need to evaluate (FUNCTION) to within (ACCURACY) using (METHOD)?"

You should have the formulas for the error bounds written down, so the main problem will be to put a bound on the derivatives of the function you want to integrate.

Bounding Derivatives

It's okay to make conservative estimates; the priority is to get a correct answer as effortlessly as possible. Some useful notes:

- $|\cos(x)| \leq 1$
- $|\sin(x)| \leq 1$
- $|A \cdot B| = |A| \cdot |B|$
- $|A + B| \leq |A| + |B|$

Practice

Bound *the absolute values* of the following functions on the given intervals:

1. $\sin x + 2\cos x, x \in [2, 10]$
   $|\sin x + 2\cos x| \leq |\sin x| + 2|\cos x| \leq 3$
   Derivative: $|\cos x - 2\sin x| \leq |\cos x| + 2|\sin x| \leq 3$
2. $x^2 e^{-x}, x \in [-1, 2]$
   $|x^2 e^{-x}| \leq |x^2| \cdot |e^{-x}| \leq 4e \leq 12$
   Derivative: $|2xe^{-x} - x^2 e^{-x}| \leq 2|x||e^{-x}| + x^2|e^{-x}| \leq 4e + 4e \leq 24$
3. $x \sin x^2 + x^2 - x, x \in [-\pi, 2]$
   $|x\sin(x^2) + x^2 - x| \leq |x| \cdot |\sin(x^2)| + |x^2| + |x| \leq \pi + \pi^2 + \pi = \pi^2 + 2\pi \leq 16 + 8 = 24$
   Derivative: $|2x^2\cos(x^2) + \sin(x^2) + 2x - 1| \leq 2x^2 + 1 + 2|x| + 1 \leq 2\pi^2 + 2\pi + 2 \leq 42$
4. $e^x/x, x \in [1, 3]$
   $|e^x/x| \leq e^3/1 = e^3 \leq 27$
   Derivative: $|e^x/x - e^x/x^2| \leq e^3 + e^3 \leq 54$
5. $\cos x^2 - 1/x, x \in [1, 2]$
   $|\cos x^2 - 1/x| \leq |\cos x^2| + |1/x| \leq 1 + 1 = 2$
   Derivative: $|-2x\sin x^2 + 1/x^2| \leq 2|x| + 1/x^2 \leq 4 + 1 = 5$
6. $x^2 - 3x + 2\sqrt{1 + x^2}, x \in [1, 3]$
   $|x^2 - 3x + 2\sqrt{1 + x^2}| \leq |x^2| + |3x| + 2|\sqrt{1 + x^2}| \leq 9 + 9 + 2\sqrt{10} = 18 + 2\sqrt{10} \leq 18 + 8 = 26$
   Derivative: $|2x - 3 + 2x/\sqrt{1 + x^2}| \leq 2|x| + 3 + 2|x|/\sqrt{1 + x^2} \leq 6 + 3 + 6/\sqrt{2} \leq 6 + 3 + 6 = 15$

Bound the absolute values of their derivatives on the same intervals.

Using the Error Bound

Find the number of intervals necessary to evaluate each of the following integrals to within 0.001 using the Trapezoid rule. If it's not too messy, do the same with Simpson's Rule.

1. $\displaystyle\int_0^1 \sin(x^2)\, dx$
   The second derivative is $2\cos(x^2) - 4x^2\sin(x^2)$, which is bounded in absolute value by 6 on the interval [0,1]. We want $|E_T| \leq \frac{K(b-a)^3}{12n^2} \leq 0.001$ and have $K = 6$, $(b - a) = 1$, so

$$\begin{aligned} 0.001 &\geq \frac{6}{12n^2} \\ n^2 &\geq \frac{6}{12(0.001)} \\ n^2 &\geq 500 \\ n &\geq 23 \end{aligned}$$

   Therefore $n = 23$ intervals will suffice.

2. $\displaystyle\int_{-1}^{1} e^{-x^2}\,dx$

The second derivative is $e^{-x^2}(4x^2-2)$. Since $e^{-x^2} \leq 1$ for all x, this is bounded in absolute value by $|4x^2-2| \leq 4x^2+2 = 6$ (being conservative) on the interval [0,1], so $K = 6$ will suffice. Here we have $(b-a)^3 = 8$, and so want

$$\begin{aligned} 0.001 &\geq \frac{48}{12n^2} \\ n^2 &\geq 4000 \\ n &\geq 64 \end{aligned}$$

Therefore $n = 64$ intervals will suffice. If we use the fact that $\int_{-1}^{1} e^{-x^2}\,dx = 2\int_0^1 e^{-x^2}\,dx$ (since the function is even), then we could get by with a smaller number of intervals.

3. $\displaystyle\int_{1}^{2} \sqrt{1+x^3}\,dx$

The second derivative is $\frac{3x(x^3+4)}{4(x^3+1)^{3/2}}$, which is bounded in absolute value on [1,2] by $K = 9$ (plugging in 2 for the values in the numerator and 1 for the values in the denominator). $(b-a)^3 = 1$ here, so to bound the error we want

$$\begin{aligned} 0.001 &\geq \frac{9}{12n^2} \\ n^2 &\geq 750 \\ n &\geq 28 \end{aligned}$$

So $n = 28$ intervals will suffice.

4. $\displaystyle\int_{1/100}^{1} \frac{1}{x}\,dx$

The second derivative is $2/x^3$, which is bounded above by $K = 2000000$. We will use $(b-a) \approx 1$ since this will be slightly easier to compute than the actual value of $(b-a)$. This is okay since the approximate value is larger and so will give a more conservative estimate. From there we get

$$\begin{aligned} 0.001 &\geq 2 \cdot 10^6/12n^2 \\ n^2 &\geq 3.33 \cdot 10^8 \\ n &\geq 1.8 \cot 10^4 \end{aligned}$$

Because the function behaves badly near $x = 0$, we have to use a large number of rectangles to get a good approximation.

5. $\displaystyle\int_{2}^{3} \sin(1/x)\,dx$

The second derivative is $\frac{2x\cos(1/x)-\sin(1/x)}{x^4}$, which is bounded by 7/8 on $[2,3]$. To make our life simpler choose $K = 1$. Then

$$\begin{aligned} 0.001 &\geq 1/6n^2 \\ n^2 &\geq 167 \\ n &\geq 14 \end{aligned}$$

So $n = 14$ will do.

6. $\int_{-1}^{1} \sin(x)e^{x^2}\,dx$

   The second derivative is $e^{x^2}(4x^2\sin(x)+\sin(x)+4x\cos(x))$, which on the interval [-1,1] is bounded in absolute value by $e(4+1+4)=9e\le 30$. Then

$$0.001 \ge 30*8/12n^2$$
$$n^2 \ge 20,000$$
$$n \ge 150$$

Bonus

1. Use the Trapezoid rule, the Midpoint rule, and Simpson's rule to estimate the value of $\int_0^1 Ax^2+Bx+C$ with $n=1$ ($n=2$ for Simpson's rule). What are the errors in each case?

   The actual integral is $A/3+B/2+C$. The Trapezoid Rule gives $A/2+B/2+C$, for an error of $A/6$. The Midpoint Rule gives $A/4+B/2+C$, for an error of $A/12$. Simpson's rule gives $A/3+B/2+C$, which is the exact answer.

2. Show that Simpson's rule with $n=2$ gets the exact value of the integral $\int_0^1 Ax^3+Bx^2+Cx+D$.

3. What is the exact value of the integral $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$? Which numerical methods overestimate it and which ones underestimate it? Use this knowledge to put upper and lower bounds on the value of π.

4. What is the exact value of the integral $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$? Use this knowledge to put upper and lower bounds on the value of π.

5. Show that $0 < \displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = 22/7-\pi$. Combine this with the fact that

$$\int_0^1 \frac{x^4(1-x)^4}{2}\,dx < \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx < \int_0^1 x^4(1-x)^4\,dx$$

   (why?) to put upper and lower bounds on the value of π. Can you beat these bounds by using numerical methods?