Numerical Integration

Wednesday, February 4

Trig

1. $\sin^2\theta + \cos^2\theta$
2. $\tan^2\theta + 1$
3. $\sec^2\theta - 1$
4. $\sec\theta\cos\theta$
5. $\tan\theta\cos\theta$
6. $\tan\theta/\sec\theta$
7. $x = \sin\theta; \tan\theta = ?$
8. $x = \tan\theta; \sec\theta = ?$
9. $x = \tan\theta; \sin\theta = ?$
10. $\int 1/\sec\theta \, d\theta$
11. $\int 1/\cos^2\theta \, d\theta$
12. $\int \tan\theta\cos\theta \, d\theta$

arctan(x), ln(x)

1. $\int \frac{3x+5}{x^2+1}$
2. $\int \frac{4x-1}{x^2+1}$
3. $\int \frac{7x+2}{x^2+1}$
4. $\int \frac{2x-4}{x^2+1}$
5. $\int \frac{x^2+2x+2}{x^2+1}$
6. $\int \frac{x^2+3x-5}{x^2+1}$

Repeated Roots

1. $\int \frac{x^2+3x+2}{x^2}$
2. $\int \frac{x^2-x+4}{x^2}$
3. $\int \frac{x^2+5}{(x-1)^2}$
4. $\int \left(\frac{x}{x-2}\right)^2$
5. $\int \frac{3x^2+4x+7}{(x+1)^2}$
6. $\int \frac{5x^2+5x+1}{(x+1)^2}$

Numerical Integration

The template for pretty much every problem, according to practice midterms:

"How many intervals do you need to evaluate (FUNCTION) to within (ACCURACY) using (METHOD)?"

You should have the formulas for the error bounds written down, so the main problem will be to put a bound on the derivatives of the function you want to integrate.

Bounding Derivatives

It's okay to make conservative estimates; the priority is to get a correct answer as effortlessly as possible. Some useful notes:

- $|\cos(x)| \leq 1$
- $|\sin(x)| \leq 1$
- $|A \cdot B| = |A| \cdot |B|$
- $|A + B| \leq |A| + |B|$

Practice

Bound *the absolute values* of the following functions on the given intervals:

1. $\sin x + 2\cos x, x \in [2, 10]$
2. $x^2 e^{-x}, x \in [-1, 2]$
3. $x \sin x^2 + x^2 - x, x \in [-\pi, 2]$
4. $e^x/x, x \in [1, 3]$
5. $\cos x^2 - 1/x, x \in [1, 2]$
6. $x^2 - 3x + 2\sqrt{1 + x^2}, x \in [1, 3]$

Bound the absolute values of their derivatives on the same intervals.

Using the Error Bound

Find the number of intervals necessary to evaluate each of the following integrals to within 0.001 using the Trapezoid rule. If it's not too messy, do the same with Simpson's Rule.

1. $\int_0^1 \sin(x^2)\, dx$
2. $\int_{-1}^1 e^{-x^2}\, dx$
3. $\int_1^2 \sqrt{1 + x^3}\, dx$
4. $\int_{1/100}^1 \frac{1}{x}\, dx$
5. $\int_2^3 \sin(1/x)\, dx$
6. $\int_{-1}^1 \sin(x) e^{x^2}\, dx$

Bonus

1. Use the Trapezoid rule, the Midpoint rule, and Simpson's rule to estimate the value of $\int_0^1 Ax^2 + Bx + C$ with $n = 1$ ($n = 2$ for Simpson's rule). What are the errors in each case?
2. Show that Simpson's rule with $n = 2$ gets the exact value of the integral $\int_0^1 Ax^3 + Bx^2 + Cx + D$.
3. What is the exact value of the integral $\displaystyle\int_0^1 \sqrt{1 - x^2}\, dx$? Which numerical methods overestimate it and which ones underestimate it? Use this knowledge to put upper and lower bounds on the value of π.
4. What is the exact value of the integral $\displaystyle\int_0^1 \frac{1}{1 + x^2}\, dx$? Use this knowledge to put upper and lower bounds on the value of π.
5. Show that $0 < \displaystyle\int_0^1 \frac{x^4(1 - x)^4}{1 + x^2}\, dx = 22/7 - \pi$. Combine this with the fact that

$$\int_0^1 \frac{x^4(1 - x)^4}{2}\, dx < \int_0^1 \frac{x^4(1 - x)^4}{1 + x^2}\, dx < \int_0^1 x^4(1 - x)^4\, dx$$

(why?) to put upper and lower bounds on the value of π. Can you beat these bounds by using numerical methods?