17.2: Nonhomogeneous Linear Equations: Solutions

Friday, April 17

Recap: Second-order Linear

Find solutions to the following differential equations:

1. $y'' + 2y' = 0; y(0) = 0, y'(0) = 3$

$$r = 0, -2$$
$$y = c_1 + c_2 e^{-2x}$$
$$y = \frac{3}{2} - \frac{3}{2}e^{-2x}$$

2. $y'' + 2y' + 3y = 0; y(0) = -1, y'(0) = 2$

$$r = -1 \pm i\sqrt{2}$$
$$y = e^{-x}(A\sin(\sqrt{2}x) + B\cos(\sqrt{2}x))$$
$$y = \frac{1}{2}e^{-x}(\sqrt{2}\sin(\sqrt{2}x) - 2\cos(\sqrt{2}x))$$

3. $y'' + 2y' + y = 0; y(0) = 3, y'(0) = 0$

$$r = -1$$
$$y = c_1 e^{-x} + c_2 x e^{-x}$$
$$y = 3e^{-x} + 3xe^{-x}$$

4. $y'' - 4y' + 9y = 0; y(0) = 1, y(3) = 4$

$$r = 2 \pm i\sqrt{5}$$
$$y = e^2(A\cos(\sqrt{5}x) + B\sin(\sqrt{5}x))$$

$$A = 1/e^2, B = \frac{4 - \cos(3\sqrt{5})}{e^2\sin(3\sqrt{5})}$$

5. $y'' - 4y' + 4y = 0; y(0) = y(2) = 0$

$$r = 2$$
$$y = c_1 e^{2x} + c_2 x e^{2x}$$
$$y = 0$$

6. $y'' - 4y' - 5y = 0; y(0) = 1, y(1) = 2$

$$r = -1, 5$$
$$y = c_1 e^{-x} + c_2 e^{5x}$$
$$c_1 = \frac{e^6 - 2e}{e^6 - 1}, c_2 = \frac{2e - 1}{e^6 - 1}$$

Nonhomogeneous Linear Equations

Let y_p be a solution to $ay'' + by' + cy = G(x)$, where a, b, c are constants.

1. If y_c solves $ay'' + by' + cy = 0$, show that $y_p + y_c$ is another solution to the original problem.
2. If y_q is another solution to $ay'' + by' + cy = G(x)$, show that $y_p - y_q$ is a solution to $ay'' + by' + cy = 0$.
3. If y_1 solves $ay'' + by' + cy = G(x)$ and y_2 solves $ay'' + by' + cy = H(x)$, show that $y_1 + y_2$ solves $ay'' + by' + cy = G(x) + H(x)$.

So to solve an equation of the form $ay'' + by' + cy = G(x) + H(x)$,

1. Find any y_g that solves $ay'' + by' + cy = G(x)$.
2. Find any y_h that solves $ay'' + by' + cy = H(x)$.
3. Find all y_c that solves $ay'' + by' + cy = 0$.
4. All solutions will be of the form $y_g + y_h + y_c$, for some y_c.

More advice! This is from the table on page 1153 of Stewart:

1. If $G(x) = e^{kx}P(x)$, try $y_p = e^{kx}Q(x)$, where $deg(P) = deg(Q)$.
2. If $G(x) = e^{kx}P(x)\cos(mx)$ or $e^{kx}P(x)\sin(mx)$, try $y_p(x) = e^{kx}Q(x)\cos(mx) + e^{kx}R(x)\sin(mx)$.
3. If you do the above and get a solution to the complementary equation, try mulitplying y_p by x or x^2.

Exercises

Find solutions to the following differential equations (the homogenous equations are the same as the ones at the start of the worksheet):

1. $y'' + 2y' = x^2 + 1$
   Try $y = Ax^2 + Bx + C$, get
   $$4Ax + (2A + 2B) = x^2 + 1$$
   This doesn't work, so try $y = Ax^3 + Bx^2 + Cx + D$. This gives $A = 1/6, B = -1/4, C = 3/4$, and any value for D. So $y_p = \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{3}{4}x$. Combining with the solution to the complementary equation gives
   $$y = \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{3}{4}x + c_1 + c_2e^{-2x}$$
2. $y'' + 2y' + 3y = 3e^{2x}$
   Try $y = ke^{2x}$, and get $4y + 4y + 3y = 11y = 11ke^{2x} = 3e^{2x}$, so $k = 3/11$ and $y_p = \frac{3}{11}e^{2x}$. Combining with the solution to the complementary equation gives
   $$y = \frac{3}{11}e^{2x} + e^{-x}(A\sin(\sqrt{2}x) + B\cos(\sqrt{2}x))$$
3. $y'' + 2y' + y = 2\sin x$
   Try $y = A\sin x + B\cos x$, and (since $y'' + y = 0$ for $\sin x$ and $\cos x$) get $2A\cos x - 2B\sin x = 2\sin x$, so $y_p = -\cos x$. Combining with the solution to the complementary equation gives
   $$y = -\cos x + c_1e^{-x} + c_2xe^{-x}$$

4. $y'' - 4y' + 9y = xe^x$

Try $y = Axe^x + Be^x$, get $y' = Ae^x + Axe^x + Be^x$, $y'' = 2Ae^x + Axe^x + Be^x$, and from there get

$$\begin{aligned} 6Axe^x + (6B - 2A)e^x &= xe^x \\ A &= 1/6 \\ B &= 1/18 \end{aligned}$$

Combining with the solution to the complementary equation gives

$$y = \frac{1}{6}Axe^x - \frac{1}{18}e^x + \cos(\sqrt{5}x) + \frac{4 - \cos(3\sqrt{5})}{\sin(3\sqrt{5})}\sin(\sqrt{5}x))$$

5. $y'' - 4y' + 4y = x^2 + e^x$

First solve $y'' - 4y' + 4y = x^2$: guess $y_1 = Ax^2 + Bx + C$ and get $y_1 = x^2/4 + x/2 + 3/8$. Then do the same for $y'' - 4y' + 4y = e^x$: guess $y_2 = Ae^x$ and get $y_2 = e^x$. Thus $y_p = e^x + x^2/4 + x/2 + 3/8$. Combining with the solution to the complementary equation gives

$$y = e^x + x^2/4 + x/2 + 3/8 + c_1e^{2x} + c_2xe^{2x}$$

6. $y'' - 4y' - 5y = x + \cos 2x$

First solve $y'' - 4y' - 5y = x$: guess $y = Ax + B$ and get $y = \frac{-1}{5}x + \frac{4}{25}$. Then do the same for $\cos 2x$ guessing $y = A\sin 2x + B\cos 2x$ and get $y = \frac{-8}{145}\sin(2x) - \frac{9}{145}\cos(2x)$. Therefore one solution is $y_p = \frac{-8}{145}\sin(2x) - \frac{9}{145}\cos(2x) - \frac{1}{5}x + \frac{4}{25}$, and the general solution is

$$y = c_1e^{-x} + c_2e^{5x} - \frac{8}{145}\sin(2x) - \frac{9}{145}\cos(2x) - \frac{1}{5}x + \frac{4}{25}$$

7. $y'' - 3y = e^{2x}\sin x$; $y(0) = 2, y'(0) = 0$

First guess $y = Ae^{2x}\sin x + Be^{2x}\cos x$ and get $y_p = \frac{-1}{4}e^{2x}\cos x$. Combine with the solutions to $y'' - 3y = 0$ and get

$$y = c_1e^{x\sqrt{3}} + c_2e^{-x\sqrt{3}} - \frac{1}{4}e^{2x}\cos x$$

Plugging in the initial conditions give the extra equations

$$\begin{aligned} c_1 + c_2 - 1/4 &= 2 \\ c_1\sqrt{3} - c_2\sqrt{3} - 1/2 &= 0 \end{aligned}$$

Solving yields the coefficients

$$c_1, c_2 = 9/8 \pm \frac{1}{4\sqrt{3}}$$

8. $y'' - y' = x^2 + x - 1$; $y(0) = 1, y(2) = 1$

Guessing $y = Ax^3 + Bx^2 + Cx + D$ and combining with the solutions to $y'' - y' = 0$ gives $y_p = c_1e^x + c_2 - x^3/3 - 3x^2/2 - 2x$. Plugging in the initial conditions gives

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1e^2 + c_2 - 2/3 &= 1 \end{aligned}$$

Solving gives the coefficients

$$c_1 = \frac{2}{3(e^2-1)}$$
$$c_2 = 1 - \frac{2}{3(e^2-1)}$$

Some More Complex Numbers

Using the formula $e^{i\theta} = \cos\theta + i\sin\theta$, write the following numbers in the form $a + bi$ and plot them on the complex plane.

1. $e^{i\pi/2} = i$
2. $e^{i\pi/4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$
3. $2e^{-i\pi/3} = \frac{1}{2} - i\sqrt{3}$
4. $\frac{1}{3}e^{i\pi} = -1/3$
5. $e^{2+i\pi/6} = e^2(\sqrt{3}/2 + i/2) = e^2\frac{\sqrt{3}}{2} + \frac{e^2}{2}i$
6. $e^{-1-i\pi} = -1/e$

1. If $z = re^{i\theta}$, then write z^2, z^3, and $1/z$ in polar form.

$$z^2 = r^2 e^{2i\theta}$$
$$z^3 = r^3 e^{3i\theta}$$
$$1/2 = \frac{1}{r}e^{-i\theta}$$

2. Find all solutions to $z^6 = 1$ by putting z in the form $re^{i\theta}$. Plot them in the complex plane.

$$z^6 = 1$$
$$(re^{i\theta})^6 = 1$$
$$r^6(\cos(6\theta) + i\sin(6\theta)) = 1$$

   The solutions are all with $r = 1$, $\theta = 0, \pi/6, 2\pi/6, 3\pi/6, 4\pi/6, 5\pi/6$. When plotted in the complex plane these six solutions will form the vertices of a regular hexagon.

3. Define the absolute value of $z = a + bi$ as $|z| = \sqrt{a^2 + b^2}$. If $z = re^{i\theta}$, then find $|z|$, $|z^2|$, and $|1/z|$.
   $|z| = r, |z^2| = r^2, |1/z| = 1/r$. This is because $|e^{i\theta}| = |cos\theta + i\sin\theta| = \cos^2\theta + \sin^2\theta = 1$ for all θ. The geometric interpretation is that for any θ the point $z = e^{i\theta}$ lies on a circle with radius 1 centered on the origin.