# 17.2: Nonhomogeneous Equations, continued

Wednesday, April 22

## Terminal Velocity

An object is dropped (with zero initial velocity) from the top of the Campanile (94 meters). The object's height $y$ as a function of the time $t$ since it was dropped can be expressed by the following differential equation:

$$y''(t) + \frac{g}{v_t} y'(t) = g$$

Here, $g$ is the acceleration due to gravity near the Earth's surface. $v_t$ is the terminal velocity of the object– the velocity at which the forces of gravity and air resistance cancel each other out. We will set "up" as the positive direction and "down" as the negative, so both $g$ and $v_t$ are negative constants.

1. If we set $g = -10m/s^2$ and $v_t = -30m/s^2$, the equation becomes $y'' + \frac{1}{3}y' = -10$. Given the intial conditions from the first paragraph, solve for $y(t)$.

   Use the method of undetermined coefficients:

$$\begin{aligned} y &= At + B \\ y' &= A \\ y'' &= 0 \\ y'' + y'/3 &= A/3 \\ A &= -30 \end{aligned}$$

   $B$ can be any constant since it disappears after taking derivatives, so $B = 0$ will suffice. A particular solution is therefor $y_p = -30t$.

   The auxiliarly equation for the homogeneous case is $r^2 + r/3 = 0$, with solutions $r = 0, -1/3$. The general solution is therefore $y = C_1 + C_2 e^{-t/3} - 30t$.

   The initial conditions are $y(0) = 94, y'(0) = 0$. Together these give

$$\begin{aligned} C_1 + C_2 &= 94 \\ -3C_2 - 30 &= 0 \\ C_2 &= -90 \\ C_1 &= 184 \end{aligned}$$

   The specific model for the height of the falling object over time is therefore

$$y = 184 - 90e^{-3t} - 30t = 94 + 90(1 - e^{-t/3}) - 30t$$

2. Do the same, but keeping the constants $g$ and $v_t$ in place of the values $-10$ and $-30$.

   Doing the same will give the solution $y = 94 - v_t^2/g(1 - e^{-gt/v_t}) - v_t t$. Note that with the units for $v_t$ being meters/second and the units for $g$ being meters/$s^2$, all of the terms on the right hand side are measured in meters, which is appropriate for the scenario. In addition the exponent for $e$ is unitless, which is also appropriate.

3. What is the velocity of the object ($y'$) as a function of $t$?

   $y' = 30e^{-3t} - 30$ (or, $y' = v_t(1 - e^{-gt/v_t})$). This implies that as $t \to \infty$, $y' \to v_t$.

4. Suppose the object were thrown downward with an initial velocity of $-40m/s$. What would the new solution for $y(t)$ look like? Describe what happens to the book in qualitative terms.

   The general solution would be the same as before, but we would get the new system of equations

$$\begin{aligned} C_1 + C_2 &= 94 \\ -3C_2 - 30 &= -40 \\ C_2 &= 10/3 \\ C_1 &= 94 - 10/3 \end{aligned}$$

   The specific solution would therefore be

$$y = 94 + \frac{10}{3}(1 - e^{-t/3}) - 30t.$$

   The object's velocity would still approach $v_t$ as $t \to \infty$, but in this case it would start out moving faster than terminal velocity and slow down over time.

## The Method of Undetermined Coefficients

Determine the form of the trial solution $y_p$ to the following differential equations:

1. $y'' + y' + 3y = x^3 + x - 1$

$$y = Ax^3 + Bx^2 + Cx + D$$

2. $y'' + 3y' = \sin 2x$

$$y = A \sin 2x + B \cos 2x$$

3. $y'' + 3y' + 2y = e^{5x}$

$$y = ke^{5x}$$

4. $y'' + 2y' + y = (x + 1) \sin 3x$

$$y = (Ax + B) \sin 3x + (Cx + D) \cos 3x$$

5. $y'' + y' + 3y = e^{2x} \cos 3x$

$$y = Ae^{2x} \cos 3x + Be^{2x} \sin 3x$$

6. $y'' + 2y' - 3y = xe^x \sin 2x$

$$y = (Ax + B)e^x \sin 2x + (Cx + D)e^x \cos 2x$$

Sometimes you might inadvertently get a solution to the complementary equation instead. Try multiplying by $x$ or $x^2$...

1. $y'' + 2y' = x^2 + 3$

$$\begin{aligned} y &= Ax^3 + Bx^2 + Cx + D \\ y_p &= x^3/6 - x^2/4 + 7x/4 + D \\ y &= x^3/6 - x^2/4 + 7x/4 + C_1 + C_2 e^{-2x} \end{aligned}$$

2. $y'' + 3y' = 4$

$$\begin{aligned} y &= Ax + B \\ y_p &= 4/3x + B \\ y &= 4/3x + C_1 + C_2 e^{-3x} \end{aligned}$$

3. $y'' - 3y' + 2y = e^x$

$$\begin{aligned} y &= Axe^x \\ y_p &= xe^x \\ y &= xe^x + C_1 + C_2 e^x \end{aligned}$$

4. $y'' + 4y = \sin 2x$

$$\begin{aligned} y &= Ax\sin 2x + Bx\cos 2x \\ y_p &= -x\cos(2x)/4 \\ y &= c_1\sin(2x) + c_2\cos(2x) - x\cos(2x)/4 \end{aligned}$$

5. $y'' = 1$

$$\begin{aligned} y'' &= Ax^2 \\ y_p &= x^2/2 \\ y &= x^2/2 + Bx + C \end{aligned}$$

6. $y'' + 2y' + y = e^{-x}$

   Since $e^{-x}$ and $xe^{-x}$ are both solutions to the homogeneous case ($r = -1$ is a double root of $r^2+2r+1$), try multiplying by $x^2$:

$$\begin{aligned} y &= kx^2e^{-x} \\ y_p &= x^2e^{-x}/2 \\ y &= x^2e^{-x}/2 + C_1e^{-x} + C_2xe^{-x} \end{aligned}$$

Find the general solutions for each of the previous six problems. For problem #$N$, find the unique solution with the initial conditions $y(0) = 1$, $y'(0) = N$.