Midterm Review 1

Wednesday, February 11

Example Functions

When deciding whether a statement is true or trying to find a counterexample, the following functions may come in handy:

1. $1/x, 1/x^2, 1/\sqrt{x}$. Or any $1/x^p$, really, but these three are the simplest.
2. $f(x) = 0$. It's always zero.
3. $f(x) = 1$. It's always one.
4. $f(x) = e^{-x}$. Helpful since $\int_1^\infty x^n e^{-x}\,dx$ converges for any n.
5. Piecewise functions: If you want a positive function that satisfies $\lim_{x\to 0} f(x) = \infty$ but where $\int_0^\infty f(x)\,dx$ converges, you could try

$$f(x) = \begin{cases} 1/\sqrt{x} & x \in [0,1] \\ 1/x^2 & x \in [1,\infty) \end{cases}$$

For each of the following, assert that is true or find a counterexample. Assume that $f(x), g(x) \geq 0$ in all cases.

1. If $\int_1^\infty xf(x)\,dx$ converges, then $\int_1^\infty f(x)\,dx$ converges.
2. If $\int_1^\infty f(x)\,dx$ converges, then $\int_1^\infty xf(x)\,dx$ converges.
3. If $\int_0^1 f(x)\,dx$ diverges, then $\int_0^1 xf(x)\,dx$ diverges.
4. If $\int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)\,dx$ converge, then $\int_1^\infty f(x) + g(x)$ converges.
5. If $\int_1^\infty f(x)\,dx$ diverges, and $\int_1^\infty g(x)\,dx$ converges, then $\int_1^\infty f(x)g(x)\,dx$ diverges.
6. If $\int_0^\infty f(x)\,dx$ always diverges.
7. If $\int_0^1 xf(x)\,dx$ diverges, then $\int_0^1 f^2(x)\,dx$ diverges.
8. At least one of $\int_0^1 f(x)\,dx$ and $\int_0^1 1/f(x)\,dx$ will always diverge.
9. At least one of $\int_1^\infty f(x)\,dx$ and $\int_1^\infty 1/f(x)\,dx$ will always diverge.
10. For every $f(x)$, there is a $g(x)$ such that $\int_1^\infty f(x) - g(x)\,dx$ converges.
11. For every $f(x)$, there is a $g(x)$ such that $\int_1^\infty f(x)g(x)\,dx$ converges.
12. If $\int_0^1 f(x)/\sqrt{x}$ diverges, then $f(x)$ is unbounded on $[0,1]$ (that is, it has a vertical asymptote somewhere).
13. If $\int_0^1 f(x)/x$ diverges, then $f(x)$ is unbounded on $[0,1]$.

Counting the Powers

Decide whether the following integrals converge or diverge:

1. $\displaystyle\int_{10}^{\infty} \frac{x^{1/2}(x+3)^{2/3}}{(x-5)^2}\,dx$
2. $\displaystyle\int_{10}^{\infty} \frac{(x+3)^4+\sin(3x)+(x-2)^2}{x^3(x-2)^3}\,dx$
3. $\displaystyle\int_{10}^{\infty} \frac{(x+\sqrt{x})^5}{(\sqrt{x}+1)^7(x-2)^2}\,dx$
4. $\displaystyle\int_{10}^{\infty} \frac{x(x+1)(x+2)(x+3)}{(x+1/x+\sin(x))^6+\sin(\sin(x))}\,dx$

Counting the Zeros/L'Hospital's Rule

Decide whether the following integrals converge or diverge:

1. $\displaystyle\int_{0}^{4} \frac{1-\cos(x)}{x^2}\,dx$
2. $\displaystyle\int_{0}^{4} \frac{\sin(x)\ln(1+x)}{x^{8/3}}\,dx$
3. $\displaystyle\int_{0}^{4} \frac{x^2-4}{x-2}\,dx$
4. $\displaystyle\int_{0}^{4} \frac{\cos(x)+1}{x-\pi}\,dx$
5. $\displaystyle\int_{0}^{4} \frac{\cos(x)+1}{(x-\pi)^2}\,dx$
6. $\displaystyle\int_{0}^{4} \frac{e^{x^2}-1}{x^{5/2}}\,dx$