Example Functions

When deciding whether a statement is true or trying to find a counterexample, the following functions may come in handy:

1. \(1/x, 1/x^2, 1/\sqrt{x}\). Or any \(1/x^p\), really, but these three are the simplest.
2. \(f(x) = 0\). It’s always zero.
3. \(f(x) = 1\). It’s always one.
4. \(f(x) = e^{-x}\). Helpful since \(\int_1^\infty x^n e^{-x} \, dx\) converges for any \(n\).
5. Piecewise functions: If you want a positive function that satisfies \(\lim_{x \to 0} f(x) = \infty\) but where \(\int_0^\infty f(x) \, dx\) converges, you could try
   \[
   f(x) = \begin{cases} 
   1/\sqrt{x} & x \in [0, 1] \\
   1/x^2 & x \in [1, \infty) 
   \end{cases}
   \]

For each of the following, assert that it is true or find a counterexample. Assume that \(f(x), g(x) \geq 0\) in all cases.

1. If \(\int_1^\infty xf(x) \, dx\) converges, then \(\int_1^\infty f(x) \, dx\) converges.
2. If \(\int_1^\infty f(x) \, dx\) converges, then \(\int_1^\infty xf(x) \, dx\) converges.
3. If \(\int_0^1 f(x) \, dx\) diverges, then \(\int_0^1 xf(x) \, dx\) diverges.
4. If \(\int_1^\infty f(x) \, dx\) and \(\int_1^\infty g(x) \, dx\) converge, then \(\int_1^\infty f(x) + g(x) \, dx\) converges.
5. If \(\int_1^\infty f(x) \, dx\) diverges, and \(\int_1^\infty g(x) \, dx\) converges, then \(\int_1^\infty f(x)g(x) \, dx\) diverges.
6. If \(\int_0^\infty f(x) \, dx\) always diverges.
7. If \(\int_0^1 xf(x) \, dx\) diverges, then \(\int_0^1 f^2(x) \, dx\) diverges.
8. At least one of \(\int_0^1 f(x) \, dx\) and \(\int_0^1 1/f(x) \, dx\) will always diverge.
9. At least one of \(\int_1^\infty f(x) \, dx\) and \(\int_1^\infty 1/f(x) \, dx\) will always diverge.
10. For every \(f(x)\), there is a \(g(x)\) such that \(\int_1^\infty f(x) - g(x) \, dx\) converges.
11. For every \(f(x)\), there is a \(g(x)\) such that \(\int_1^\infty f(x)g(x) \, dx\) converges.
12. If \(\int_0^1 f(x)/\sqrt{x}\) diverges, then \(f(x)\) is unbounded on \([0, 1]\) (that is, it has a vertical asymptote somewhere).
13. If \(\int_0^1 f(x)/x\) diverges, then \(f(x)\) is unbounded on \([0, 1]\).
Counting the Powers

Decide whether the following integrals converge or diverge:

1. \[ \int_{10}^{\infty} \frac{x^{1/2}(x + 3)^{2/3}}{(x - 5)^2} \, dx \]
2. \[ \int_{10}^{\infty} \frac{(x + 3)^4 + \sin(3x) + (x - 2)^2}{x^3(x - 2)^3} \, dx \]
3. \[ \int_{10}^{\infty} \frac{(x + \sqrt{x})^5}{(\sqrt{x} + 1)^7(x - 2)^2} \, dx \]
4. \[ \int_{10}^{\infty} \frac{x(x + 1)(x + 2)(x + 3)}{(x + 1/x + \sin(x))^6 + \sin(\sin(x))} \, dx \]

Counting the Zeros/L’Hospital’s Rule

Decide whether the following integrals converge or diverge:

1. \[ \int_{0}^{4} \frac{1 - \cos(x)}{x^2} \, dx \]
2. \[ \int_{0}^{4} \frac{\sin(x) \ln(1 + x)}{x^{8/3}} \, dx \]
3. \[ \int_{0}^{4} \frac{x^2 - 4}{x - 2} \, dx \]
4. \[ \int_{0}^{4} \frac{\cos(x) + 1}{x - \pi} \, dx \]
5. \[ \int_{0}^{4} \frac{\cos(x) + 1}{(x - \pi)^2} \, dx \]
6. \[ \int_{0}^{4} \frac{e^{x^2} - 1}{x^{5/2}} \, dx \]