Midterm Review 1–Solutions

Monday, February 9

Example Functions

When deciding whether a statement is true or trying to find a counterexample, the following functions may come in handy:

1. $1/x, 1/x^2, 1/\sqrt{x}$. Or any $1/x^p$, really, but these three are the simplest.
2. $f(x) = 0$. It's always zero.
3. $f(x) = 1$. It's always one.
4. $f(x) = e^{-x}$. Helpful since $\int_1^\infty x^n e^{-x}\,dx$ converges for any n.
5. Piecewise functions: If you want a positive function that satisfies $\lim_{x\to 0} f(x) = \infty$ but where $\int_0^\infty f(x)\,dx$ converges, you could try

$$f(x) = \begin{cases} 1/\sqrt{x} & x \in [0,1] \\ 1/x^2 & x \in [1,\infty) \end{cases}$$

For each of the following, assert that is true or find a counterexample. Assume that $f(x), g(x) \geq 0$ in all cases.

1. If $\int_1^\infty xf(x)\,dx$ converges, then $\int_1^\infty f(x)\,dx$ converges.
   True by the comparison test, since $xf(x) \geq f(x)$ when $x \geq 1$.
2. If $\int_1^\infty f(x)\,dx$ converges, then $\int_1^\infty xf(x)\,dx$ converges.
   False: $f(x) = 1/x^2$.
3. If $\int_0^1 f(x)\,dx$ diverges, then $\int_0^1 xf(x)\,dx$ diverges.
   False: $f(x) = 1/x$.
4. If $\int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)\,dx$ converge, then $\int_1^\infty f(x) + g(x)$ converges.
   True.
5. If $\int_1^\infty f(x)\,dx$ diverges, and $\int_1^\infty g(x)\,dx$ converges, then $\int_1^\infty f(x)g(x)\,dx$ diverges.
   False: $f(x) = 1/x, g(x) = 1/x^2$, or just $g(x) = 0$ will do.
6. If $\int_0^\infty f(x)\,dx$ always diverges.
   False: $f(x) = e^{-x}, f(x) = 0, f(x) = \begin{cases} 1 & x \leq 1 \\ 1/x^2 & x \geq 1 \end{cases}$
7. If $\int_0^1 xf(x)\,dx$ diverges, then $\int_0^1 f^2(x)\,dx$ diverges.
   True: $xf(x) \leq f(x)$ for $0 \leq x \leq 1$, so $\int_0^1 f(x)$ diverges. This in turn means that $\int_0^1 f^2(x)$ diverges, though the proof is a little more subtle: one way to do it is to set $g(x) = \begin{cases} f(x) & f(x) \geq 1 \\ 0 & f(x) < 1, \end{cases}$

   in which case $\int_0^1 f^2(x) \geq \int_0^1 g^2(x) \geq \int_0^1 g(x) \geq \int_0^1 f(x) - 1$. Since $\int_0^1 f(x)\,dx$ diverges, $\int_0^1 f^2(x)\,dx$ does too by the comparison test. Roughly, the idea behind this is that $\int f(x)$ diverges because of its vertical asymptotes, so the vertical asymptotes of $\int f^2(x)$ will be even "worse" as far as convergence is concerned since when $f(x)$ is large $f^2(x)$ will be much larger.

8. At least one of $\int_0^1 f(x)\,dx$ and $\int_0^1 1/f(x)\,dx$ will always diverge.

   False: $f(x) = 1$ or $f(x) = \sqrt{x}$ will do as counterexamples.

9. At least one of $\int_1^\infty f(x)\,dx$ and $\int_1^\infty 1/f(x)\,dx$ will always diverge.

   True, because for $\int_1^\infty f(x)\,dx$ to converge we need $\lim_{x\to\infty} f(x) = 0$, in which case $\lim_{x\to\infty} 1/f(x) = \infty$, making $\int_1^\infty 1/f(x)\,dx$ diverge.

10. For every $f(x)$, there is a $g(x)$ such that $\int_1^\infty f(x) - g(x)\,dx$ converges.

    True: let $g(x) = f(x)$.

11. For every $f(x)$, there is a $g(x)$ such that $\int_1^\infty f(x)g(x)\,dx$ converges.

    True: Let $g(x) = 0$ (if we require $g(x) > 0$, then let $g(x) = \frac{1}{x^2 f(x)}$).

12. If $\int_0^1 f(x)/\sqrt{x}$ diverges, then $f(x)$ is unbounded on $[0, 1]$ (that is, it has a vertical asymptote somewhere).

    True: Suppose $f(x) \le C$ on $[0, 1]$. Then $\int_0^1 f(x)/\sqrt{x} \le \int_0^1 C/\sqrt{x}$, which converges. This means that if $f(x)$ is bounded then $\int_0^1 f(x)/x$ converges. By the contrapositive of that statement, if $f(x)/x$ diverges then $f(x)$ must be unbounded.

13. If $\int_0^1 f(x)/x$ diverges, then $f(x)$ is unbounded on $[0, 1]$.

    False: $f(x) = 1$.

Counting the Powers

Decide whether the following integrals converge or diverge:

1. $$\int_{10}^\infty \frac{x^{1/2}(x+3)^{2/3}}{(x-5)^2}\,dx$$

   Looking at the highest powers of x gives $\displaystyle\int \frac{x^{1/2}x^{2/3}}{x^2} = \int \frac{1}{x^{5/6}}$, which diverges by the p-test.

2. $$\int_{10}^\infty \frac{(x+3)^4 + \sin(3x) + (x-2)^2}{x^3(x-2)^3}\,dx$$

   Looking at the highest powers gives $\displaystyle\int \frac{x^4}{x^6} = \int 1/x^2$, which converges.

3. $$\int_{10}^\infty \frac{(x+\sqrt{x})^5}{(\sqrt{x}+1)^7(x-2)^2}\,dx$$

   Looking at the highest powers gives $\displaystyle\int \frac{x^5}{x^{7/2}x^2} = \int 1/x^{1/2}$, which diverges.

4. $$\int_{10}^\infty \frac{x(x+1)(x+2)(x+3)}{(x+1/x+\sin(x))^6 + \sin(\sin(x))}\,dx$$

   Looking at the highest powers gives $\displaystyle\int \frac{x^4}{x^6} = \int 1/x^2$, which converges.

Counting the Zeros/L'Hospital's Rule

Decide whether the following integrals converge or diverge:

1. $\displaystyle\int_0^4 \frac{1-\cos(x)}{x^2}\,dx$

   The potential asymptote is at $x = 0$. Check by L'Hospital's rule:

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos(x)}{x^2} &= \lim_{x\to 0}\frac{\sin(x)}{2x}\\ &= \lim_{x\to 0}\frac{\cos(x)}{2}\\ &= 1/2\end{aligned}$$

   The limit is finite, so the function is bounded and the integral converges.

2. $\displaystyle\int_0^4 \frac{\sin(x)\ln(1+x)}{x^{8/3}}\,dx$

   The potential asymptote is at $x = 0$. Use the knowledge that $\lim_{x\to 0}\sin(x)/x = \lim_{x\to 0}\ln(1+x)/x = 1$ (check with L'Hop's rule) to simplify:

$$\frac{\sin(x)\ln(1+x)}{x^{8/3}} = \frac{\sin x}{x}\frac{\ln(1+x)}{x}\frac{1}{x^{2/3}} \approx \frac{1}{x^{2/3}}$$

   as $x \to 0$. The function therefore grows like $x^{2/3}$, and so the integral converges. x

3. $\displaystyle\int_0^4 \frac{x^2-4}{x-2}\,dx$

   $\frac{x^2-4}{x-2} = x+2$ when $x \neq 2$, so the function is bounded and the integral converges.

4. $\displaystyle\int_0^4 \frac{\cos(x)+1}{x-\pi}\,dx$

   The potential asymptote is at $x = \pi$. Using L'Hospital's rule gives

$$\begin{aligned}\lim_{x\to\pi}\frac{\cos(x)+1}{x-\pi} &= \lim_{x\to\pi}\frac{-\sin(x)}{1}\\ &= 0\end{aligned}$$

   So the function is bounded, and the integral converges.

5. $\displaystyle\int_0^4 \frac{\cos(x)+1}{(x-\pi)^2}\,dx$

   Same as before, but one more step of L'Hop's rule is needed.

$$\begin{aligned}\lim_{x\to\pi}\frac{\cos(x)+1}{(x-\pi)^2} &= \lim_{x\to\pi}\frac{-\sin(x)}{2(x-\pi)}\\ &= \lim_{x\to\pi}\frac{-\cos(x)}{2}\\ &= 1/2\end{aligned}$$

The function is again bounded, and the integral converges. Note that if the denominator had been $(x-\pi)^3$, or raised to any power greater than 3, then the integral would have diverges. Any power $2 < p < 3$ would give a convergent integral.

6. $\displaystyle\int_0^4 \frac{e^{x^2}-1}{x^{5/2}}\,dx$

L'Hop's rule at $x=0$, done twice in a row since the first derivative of e^{x^2} is still 0 at $x=0$:

$$\begin{aligned}
\lim_{x\to 0}\frac{e^{x^2}-1}{x^{5/2}} &= \lim_{x\to 0}\frac{2xe^{x^2}}{5/2x^{3/2}} \\
&= \lim_{x\to 0}\frac{2e^{x^2}+4x^2e^{x^2}}{15/4x^{1/2}} \\
&= \frac{8}{15}\frac{1}{x^{1/2}}
\end{aligned}$$

Since the function grows like $\frac{1}{\sqrt{x}}$ at the singularity, the integral converges.