17.1: Second-order Linear Equations

Wednesday, April 15

Recap: First-Order Linear Equations

To solve a differential equation of the form $y' + P(x)y = Q(x)\ldots$

1. Set $I(x) = e^{\int P(x)\,dx}$, noting that $I' = PI$.
2. Multiply the equation by I, getting $Iy' + IPy = IQ$.
3. Since $Iy' + IPy = (Iy)'$, this turns into $Iy = C + \int IQ$
4. Condensed solution: $y = (C + \int IQ)/I$, for any C.

Solve the linear systems:

1. $y' = x - y$
2. $y' - y = e^x$
3. $y'/x - y/x = 3, y(1) = y'(1) = -3$
4. $2xy' + y = 6x, x > 0, y(4) = 20$

Interlude: Complex Numbers

Let i be a solution to the equation $x^2 + 1 = 0$, so that $i^2 = -1$.

1. Write the product $(1 + 2i)(2 - 3i)$ in the form $a + bi$.
2. Write the product $(3 + 2i)(3 - 2i)$ in the form $a + bi$.
3. Find the two solutions to $x^2 + 3 = 0$.
4. Find the three solutions to $x^3 - 1 = 0$ (Hint: one of the roots, r, is easy to find. Then divide $x^3 - 1$ by $(x - r)$).
5. Let $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$. Find $z^2, z^3, z^4, z^5, z^6, z^7$, and z^8 and plot them all in the complex plane.

Families of Solutions

Suppose that $f(x)$ and $g(x)$ are both solutions to the differential equation $y'' - y = 0$.

1. Show that $Af(x) + Bg(x)$ is also a solution for any constants A and B.
2. Find infinitely many functions that solve $y'' = 1$.
3. Find *all* of the solutions. Can you prove that there aren't any others?

Second-order Linear Equations

To solve the system $ay'' + by' + cy = 0$, find the roots r_1, r_2 to the equation $ax^2 + bx + c = 0$.

1. If $r_1 \neq r_2$, then the general solution is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.
2. If $r_1 = r_2$, then the general solution is $y = c_1 e^{rx} + c_2 x e^{rx}$.

- Verify that these are in fact solutions.

In the event that $r_1, r_2 = \alpha \pm \beta i$, it is considered impolite to give an imaginary solution to a real differential equation. Use Euler's identity $e^{i\theta} = \cos\theta + i \sin\theta$ to find real solutions instead.
Find solutions to the following linear differential equations:

1. $y'' + y = 0$
2. $y'' - 4y' + y = 0$
3. $y'' + 4y' + 4y = 0$
4. $y'' - y' - 12y = 0, y(1) = 0, y'(1) = 1$
5. $y'' = y', y(0) = 1, y(1) = 2$
6. $2y'' + y' - y = 0, y(0) = 3, y'(0) = 3$