17.1: Second-order Linear Equations

Wednesday, April 15

Recap: First-Order Linear Equations

To solve a differential equation of the form $y' + P(x)y = Q(x)\ldots$

1. Set $I(x) = e^{\int P(x)\,dx}$, noting that $I' = PI$.
2. Multiply the equation by I, getting $Iy' + IPy = IQ$.
3. Since $Iy' + IPy = (Iy)'$, this turns into $Iy = C + \int IQ$
4. Condensed solution: $y = (C + \int IQ)/I$, for any C.

Solve the linear systems:

1. $y' = x - y$

$$\begin{aligned} y' + y &= x \\ I &= e^x \\ e^x y' + e^x y &= xe^x \\ (e^x y)' &= xe^x \\ e^x y &= xe^x - e^x + C \\ y &= x - 1 + Ce^{-x} \end{aligned}$$

2. $y' - y = e^x$

$$\begin{aligned} I &= e^{-x} \\ \int IQ &= x \\ y &= Ce^x + xe^x \end{aligned}$$

3. $y'/x - y/x = 3, y(1) = y'(1) = -3$

$$\begin{aligned} y' - y &= 3x \\ I &= e^{-x} \\ \int IQ &= 3\int xe^{-x} \\ &= -3xe^{-x} - 3e^{-x} \\ y &= e^x(C - 3e^{-x} - 3xe^{-x}) \\ &= Ce^x - 3x - 3 \\ y &= -3x - 3 \end{aligned}$$

4. $2xy' + y = 6x, x > 0, y(4) = 20$

$$\begin{aligned} y' + y/2x &= 3 \\ I &= e^{\ln x/2} \\ &= \sqrt{x} \\ \int IQ &= \int 3\sqrt{x} \\ &= 2x^{3/2} \\ y &= C/\sqrt{x} + 2x \end{aligned}$$

Interlude: Complex Numbers

Let i be a solution to the equation $x^2 + 1 = 0$, so that $i^2 = -1$.

1. Write the product $(1 + 2i)(2 - 3i)$ in the form $a + bi$.

$$(1 + 2i)(2 - 3i) = 2 - 3i + 4i - 6i^2 = 8 + i$$

2. Write the product $(3 + 2i)(3 - 2i)$ in the form $a + bi$.

$$(3 + 2i)(3 - 2i) = 9 - 6i + 6i - 4i^2 = 13$$

3. Find the two solutions to $x^2 + 3 = 0$.
   $x + \pm\sqrt{-3} = \pm i\sqrt{3}$.

4. Find the three solutions to $x^3 - 1 = 0$ (Hint: one of the roots, r, is easy to find. Then divide $x^3 - 1$ by $(x - r)$).
   $x^3 - 1 = (x - 1)(x^2 + x + 1)$, so the solutions are $x = 1$ and

$$x = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

5. Let $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$. Find $z^2, z^3, z^4, z^5, z^6, z^7$, and z^8 and plot them all in the complex plane.
   The powers of z all lie on the unit circle, going around counter-clockwise.

Families of Solutions

Suppose that $f(x)$ and $g(x)$ are both solutions to the differential equation $y'' - y = 0$.

1. Show that $Af(x) + Bg(x)$ is also a solution for any constants A and B.

$$(Af + Bg)'' - (Af + Bg) = Af'' - Af + Bg'' - Bg = 0 + 0 = 0$$

2. Find infinitely many functions that solve $y'' = 1$.

   All functions of the form $y = x^2/2 + Ax + B$ are solutions.

3. Find *all* of the solutions. Can you prove that there aren't any others?

   If $y'' - 1 = 0$, then $(y' - x)' = 0$ and so by the Mean Value Theorem $y' - x = A$. Since $y' - x - A = (y - x^2/2 - Ax)' = 0$, use the MVT again to get $y - x^2/2 - Ax = B$, and so $y = x^2/2 + Ax + B$. These solutions are therefore the only ones.

Second-order Linear Equations

To solve the system $ay'' + by' + cy = 0$, find the roots r_1, r_2 to the equation $ax^2 + bx + c = 0$.

1. If $r_1 \neq r_2$, then the general solution is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.
2. If $r_1 = r_2$, then the general solution is $y = c_1 e^{rx} + c_2 x e^{rx}$.

- Verify that these are in fact solutions.

In the event that $r_1, r_2 = \alpha \pm \beta i$, it is considered impolite to give an imaginary solution to a real differential equation. Use Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$ to find real solutions instead.
Find solutions to the following linear differential equations:

1. $y'' + y = 0$

$$y = A\sin\theta + B\cos\theta$$

2. $y'' - 4y' + y = 0$

$$y = Ae^{r_1 x} + Be^{r_2 x}, r_1, r_2 = 2 \pm \sqrt{3}$$

3. $y'' + 4y' + 4y = 0$

$$y = Ae^{-2x} + Bxe^{-2x}$$

4. $y'' - y' - 12y = 0, y(1) = 0, y'(1) = 1$

$$y = Ae^{-3x} + Be^{4x}$$

5. $y'' = y', y(0) = 1, y(1) = 2$

$$y = Ae^x + B$$
$$y = e^x/e + 1 - 1/e$$

6. $2y'' + y' - y = 0, y(0) = 3, y'(0) = 3$

$$\begin{aligned} y &= Ae^{x/2} + Be^x \\ A + B &= 3 \\ A/2 + B &= 3 \\ B &= 3 \\ y &= 3e^x \end{aligned}$$