11.2: Intro to Series

Wednesday, Febrary 19

Speed Round

1. $\ln(a) + \ln(b) =$
2. $x^{a+b} =$
3. $\lim_{x\to 0} \frac{\sin x}{x}$
4. $e^x = 1 + x + \ldots$
5. $\sin x = x - x^3/3! + \ldots$
6. $\cos x = 1 - x^2/2! + \ldots$
7. $\lim_{n\to\infty} e^n/n!$
8. $\lim_{n\to\infty} e^n/n^e$
9. $\lim_{n\to\infty} \ln^3(n)/\sqrt{n}$
10. $\lim_{n\to\infty}(-1)^n$
11. $\lim_{n\to\infty} \frac{2^n+\sin n}{3^n}$
12. $\sum_{i=1}^{3}(2i+1)$
13. $\sum_{i=1}^{4}\left(\frac{1}{2}\right)^i$
14. $\lim_{n\to\infty} \frac{3^n+n^3}{3^n-n^2+\ln n}$
15. $\lim_{n\to\infty} \frac{e^n+n^{100}}{n!}$
16. $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n$
17. $\lim_{n\to\infty} \frac{1}{n} + \frac{\sin(n)\ln(n)}{n^2}$
18. $\lim_{n\to\infty} \frac{3^n+n^2}{n^{100}+2^n}$

Geometric Series

Proof of main result:

$$\begin{aligned} S_n &= 1 + x + x^2 + \ldots + x^n \\ xS_n &= x + x^2 + \ldots + x^n + x^{n+1} \\ (1-x)S_n &= 1 - x^{n+1} \\ S_n &= \frac{1-x^{n+1}}{1-x} \\ \lim_{n\to\infty} S_n &= \frac{1}{1-x} - \lim_{n\to\infty}\frac{x^{n+1}}{1-x} \\ 1 + x + x^2 + \ldots &= \frac{1}{1-x} \text{ (Provided } |x| < 1) \end{aligned}$$

Extension, which holds for all a provided $|r| < 1$:

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

1. Find $\displaystyle\sum_{n=0}^{\infty} \frac{3}{4^n}$.
2. Find the limit of $5 - 5/2 + 5/4 - 5/8 + \ldots$

Harmonic Series

Big picture: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges even though $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$.

Fun Fact: If you added one term of the harmonic series per second starting 13.8 billion years ago, your running sum today would be about 41.2.

Test That Doesn't Merit Any Name Better Than "Test For Divergence"

If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.

Practice Problems

Decide whether the following series converge using one of the three tests above. If they converge, find the limit.

1. $\sum_{n=1}^{\infty} \frac{1}{3n}$
2. $\sum_{n=1}^{\infty} \frac{1}{3n+1}$
3. $\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$
4. $\sum_{n=1}^{\infty} \frac{5}{2^n}$
5. $\sum_{n=1}^{\infty} \ln n$
6. $\sum_{n=1}^{\infty} \frac{\ln n + 1}{\ln n}$
7. $\sum_{n=1}^{\infty} \left(\frac{2^n}{3^n} + \frac{1}{n}\right)$
8. $\sum_{n=1}^{\infty} \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^n$
9. $\sum_{n=1}^{\infty} \frac{n^2+1}{2n^2}$
10. $\sum_{n=1}^{\infty} \frac{n^2+1}{2n^3}$
11. $\sum_{n=1}^{\infty} \frac{7}{e^n}$
12. $\sum_{n=1}^{\infty} \ln \frac{2n+1}{n}$
13. $\sum_{n=1}^{\infty} \frac{1}{3^n} + \frac{1}{2^n}$

True/False

If false, find a counterexample.

1. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
2. If $\{a_n\}$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
3. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.
4. If $\lim_{n\to\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+3}$.
5. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=10}^{\infty} a_n$ converges.
6. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=10}^{\infty} a_n = \sum_{n=1}^{\infty} a_n$.
7. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} c \cdot a_n$ diverges for any $c \neq 0$.
8. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge, then $\sum_{n=1}^{\infty} a_n + b_n$ diverges.

Bonus

1. Write 1.027027027... as a rational number.
2. Write 1/33 as a repeating decimal.
3. Write $\frac{1}{0.997}$ to as many decimal places as you can.
4. Sketch the graphs of $\frac{1}{1-x}, 1 + x + x^2 + \ldots x^{50}$, and $1 + x + x^2 + \ldots + x^{51}$ (you can plug in the points $x = 0$ and $x = 1$ to help with your sketches).
5. What is $1/(1 + x)$, written as an infinite series? (Hint: $1 + x = 1 - (-x)$)