11.2: Intro to Series

Wednesday, Febrary 19

Speed Round

1. $\ln(a) + \ln(b) = \ln(ab)$
2. $x^{a+b} = x^a \cdot x^b$
3. $\lim_{x\to 0} \frac{\sin x}{x} = 1$
4. $e^x = 1 + x + x^2/2! + x^3/3! + \dots$
5. $\sin x = x - x^3/3! + x^5/5! - x^7/7! + \dots$
6. $\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \dots$
7. $\lim_{n\to\infty} e^n/n! = 0$
8. $\lim_{n\to\infty} e^n/n^e = \infty$
9. $\lim_{n\to\infty} \ln^3(n)/\sqrt{n} = 0$
10. $\lim_{n\to\infty}(-1)^n$ DNE
11. $\lim_{n\to\infty} \frac{2^n+\sin n}{3^n} = 0$
12. $\sum_{i=1}^{3}(2i+1) = 3 + 5 + 7 = 15$
13. $\sum_{i=1}^{4}\left(\frac{1}{2}\right)^i = 1/2 + 1/4 + 1/8 + 1/16 = 15/16$
14. $\lim_{n\to\infty} \frac{3^n+n^3}{3^n-n^2+\ln n} = \infty$
15. $\lim_{n\to\infty} \frac{e^n+n^{100}}{n!} = 0$
16. $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$
17. $\lim_{n\to\infty} \frac{1}{n} + \frac{\sin(n)\ln(n)}{n^2} = 0$
18. $\lim_{n\to\infty} \frac{3^n+n^2}{n^{100}+2^n} = \infty$

Geometric Series

Proof of main result:

$$\begin{aligned}
S_n &= 1 + x + x^2 + \dots + x^n \\
xS_n &= x + x^2 + \dots + x^n + x^{n+1} \\
(1-x)S_n &= 1 - x^{n+1} \\
S_n &= \frac{1-x^{n+1}}{1-x} \\
\lim_{n\to\infty} S_n &= \frac{1}{1-x} - \lim_{n\to\infty}\frac{x^{n+1}}{1-x} \\
1 + x + x^2 + \dots &= \frac{1}{1-x} \text{ (Provided } |x| < 1)
\end{aligned}$$

Extension, which holds for all a provided $|r| < 1$:

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

1. Find $\displaystyle\sum_{n=0}^{\infty} \frac{3}{4^n}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{3}{4^n} &= \frac{3}{1-1/4} \\ &= \frac{3}{3/4} \\ &= 4\end{aligned}$$

2. Find the limit of $5 - 5/2 + 5/4 - 5/8 + \ldots$

$$\begin{aligned}\lim_{n\to\infty} 5 - 5/2 + 5/4 - 5/8 \ldots &= \sum_{n=0}^{\infty} 5 \cdot (-1/2)^n \\ &= \frac{5}{1+1/2} \\ &= 10/3\end{aligned}$$

Harmonic Series

Big picture: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges even though $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$.
Fun Fact: If you added one term of the harmonic series per second starting 13.8 billion years ago, your running sum today would be about 41.2.

Test That Doesn't Merit Any Name Better Than "Test For Divergence"

If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.

Practice Problems

Decide whether the following series converge using one of the three tests above. If they converge, find the limit.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n}$ diverges (Harmonic series)

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n+1}$ diverges (This is getting ahead in the material, but $\sum_{n=1}^{\infty} \frac{1}{3n+1} > \sum_{n=1}^{\infty} \frac{1}{3n+3} = 3\sum_{n=1}^{\infty} \frac{1}{n+1}$, which diverges since it contains all but the first term of the Harmonic series.)

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$ converges.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1+2^n}{3^n} &= \sum_{n=1}^{\infty}(1/3)^n + \sum_{n=1}^{\infty}(2/3)^n \\
&= 1/3\sum_{n=1}^{\infty}(1/3)^{n-1} + 2/3\sum_{n=1}^{\infty}(2/3)^{n-1} \\
&= 1/3(\frac{1}{1-1/3}) + 2/3(\frac{1}{1-2/3}) \\
&= 1/3(3/2) + 2/3(3) \\
&= 5/2
\end{aligned}$$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2^n}$ converges (GS)

$$\sum_{n=1}^{\infty} \frac{5}{2^n} = 5/2\sum_{n=1}^{\infty}(1/2)^{n-1} = (5/2)(2) = 5$$

5. $\displaystyle\sum_{n=1}^{\infty} \ln n$ diverges ($\lim_{n\to\infty} \ln n \neq 0$)

6. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n + 1}{\ln n}$ diverges (The limit of the sequence is not zero)

7. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2^n}{3^n} + \frac{1}{n}\right)$ diverges (Because the part with the Harmonic series diverges, the entire sum diverges)

8. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^n$ converges.

$$\begin{aligned}
\sum_{n=1}^{\infty} \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^n &= \frac{\sqrt{2}}{\sqrt{3}}\sum_{n=1}^{\infty} \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^{n-1} \\
&= \frac{\sqrt{2}}{\sqrt{3}}\frac{1}{1-\frac{\sqrt{2}}{\sqrt{3}}} \\
&= \frac{\sqrt{2}}{\sqrt{3}-\sqrt{2}} \\
&= \frac{\sqrt{2}}{\sqrt{3}-\sqrt{2}}\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} \\
&= 2+\sqrt{6}
\end{aligned}$$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{2n^2}$ diverges (The limit of the sequence is not zero)

10. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{2n^3}$ diverges (Again getting ahead in the material: the function "looks like" $1/2n$ as $n \to \infty$, so the function diverges like the Harmonic series)

11. $\sum_{n=1}^{\infty} \frac{7}{e^n}$ converges

$$\sum_{n=1}^{\infty} \frac{7}{e^n} = 7/e \sum_{n=1}^{\infty} (1/e)^{n-1} = \frac{7}{e}(1 - 1/e) = 7/(e-1)$$

12. $\sum_{n=1}^{\infty} \ln \frac{2n+1}{n}$ diverges (The limit of the sequence is not zero)

13. $\sum_{n=1}^{\infty} \frac{1}{3^n} + \frac{1}{2^n}$ converges

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{3^n} + \frac{1}{2^n} &= \frac{1}{3}\sum_{n=1}^{\infty}(1/3)^{n-1} + \frac{1}{2}\sum_{n=1}^{\infty}(1/2)^{n-1} \\ &= \frac{1}{3}\frac{1}{1-1/3} + \frac{1}{2}\frac{1}{1-1/2} \\ &= 1/2 + 1 \\ &= 3/2\end{aligned}$$

True/False

If false, find a counterexample.

1. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. True.
2. If $\{a_n\}$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. False: $a_n = 1$.
3. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges. False: $a_n = 1/n$.
4. If $\lim_{n\to\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+3}$. True.
5. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=10}^{\infty} a_n$ converges. True.
6. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=10}^{\infty} a_n = \sum_{n=1}^{\infty} a_n$. False–pretty much any series will do.
7. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} c \cdot a_n$ diverges for any $c \neq 0$. True.
8. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge, then $\sum_{n=1}^{\infty} a_n + b_n$ diverges. False: $b_n = -a_n$ will give $a_n + b_n = 0$.

Bonus

1. Write 1.027027027... as a rational number.$1 + 27/999 = 1 + 1/37$
2. Write 1/33 as a repeating decimal. $0.0303030303\ldots$
3. Write $\frac{1}{0.997}$ to as many decimal places as you can. $0.0030090270812437\ldots$
4. Sketch the graphs of $\frac{1}{1-x}, 1 + x + x^2 + \ldots x^{50}$, and $1 + x + x^2 + \ldots + x^{51}$ (you can plug in the points $x = 0$ and $x = 1$ to help with your sketches). They are all very close for the range $0.05 < x < 1$.
5. What is $1/(1+x)$ written as an infinite series?

$$1/(1+x) = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$$

   This series also converges for $|x| < 1$.