11.3: The Integral Test

Wednesday, February 25

Recap: Sequences

Order the following sequences by their growth rate as $n \to \infty$:

$$n, \ln(n), \sqrt{n}, 1.01^n, \sqrt{n^3+2}, n!, 50, n^2, n^{100}, 2^n, 50^n, \ln(n)^{30}, n^{1.001}$$

Find the limits of the following sequences:

1. $\lim_{n\to\infty} \frac{n^5}{2^n}$
2. $\lim_{n\to\infty} \frac{n^5}{\ln(n)^5}$
3. $\lim_{n\to\infty} \frac{e^n}{n!}$
4. $\lim_{n\to\infty} \frac{n^{10}}{n!}$
5. $\lim_{n\to\infty} \frac{1.01^n}{\sqrt{n}}$
6. $\lim_{n\to\infty} \frac{\ln(n)}{e^n}$
7. $\lim_{n\to\infty} \frac{n^3+n^2}{\sqrt{n^7+5}}$
8. $\lim_{n\to\infty} \frac{1.2^n+n^5}{1.1^n-n^3}$
9. $\lim_{n\to\infty} \frac{e^n+\sqrt{7n+2}}{\ln(n)+n!}$
10. $\lim_{n\to\infty} \frac{n^4+\ln(n)}{3n^3+4n^2+n+1}$
11. $\lim_{n\to\infty} \frac{1.03^n+e^n}{n^7+2e^n}$
12. $\lim_{n\to\infty} \frac{\sqrt{2n^3+5}}{3n\sqrt{n-1}+\ln(n)}$

Recap: Geometric Series

Formula for the sum of a geometric series when $|r| < 1$:

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

Writing out the individual terms, we can get

$$a + ar + ar^2 + \ldots = a(1 + r + r^2 + \ldots) = \frac{a}{1-r}$$

So one way to find the sum is to write the first two terms of the sequence, then factor out the first term and use $1 + r + r^2 + \ldots = \frac{1}{1-r}$.

Example: $\sum_{n=1}^{\infty} \frac{3}{2^{n+2}} = 3/8 + 3/16 + \ldots = \frac{3}{8}(1 + 1/2 + \ldots) = \frac{3}{8}\frac{1}{1-1/2} = 3/4.$

1. $\sum_{n=1}^{\infty} 1/3^n$
2. $\sum_{n=1}^{\infty} 5/8^n$
3. $\sum_{n=0}^{\infty} 3^{n+2}/4^{n-1}$
4. $\sum_{n=0}^{\infty} 2^{n+3}/3^{n+2}$
5. $\sum_{n=1}^{\infty} 5 \cdot 2^n/3^{n+1}$
6. $\sum_{n=1}^{\infty} 7 \cdot 2^{n+2}/5^n$

The Integral Test

1. For a positive decreasing (or eventually decreasing) sequence a_n and corresponding function f, the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.
2. $\int_1^n f(x)\,dx \le \sum_{i=1}^n a_n \le a_1 + \int_1^n f(x)\,dx$.
3. If $s = \sum a_n$ and s_n is the nth partial sum, then $\displaystyle\int_{n+1}^{\infty} f(x)\,dx \le R_n = s - s_n \le \int_n^{\infty} f(x)\,dx$.

Example: Since $a_n = 1/n$ is decreasing and $\int_1^{\infty} \frac{1}{x}\,dx$ diverges, the harmonic series diverges.
Decide whether the following series are convergent or divergent by using the integral test:

1. $\displaystyle\sum_{n=1}^{\infty} 1/n$
2. $\displaystyle\sum_{n=1}^{\infty} 1/n^2$
3. $\displaystyle\sum_{n=1}^{\infty} 1/\sqrt{n}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\ln(n)}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\ln(n)^2}$
6. $\displaystyle\sum_{n=1}^{\infty} 1/n^3$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{1+n^2}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$

Decide whether the followng integrals are convergent or divergent by using the integral test. You do not have to compute the integral.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+\sqrt{n}}{n^3+\ln n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^6-n^5}{n^4+n^3+\sin(n)}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^3}{n^5+7}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^3-n^2+n}{n^2+\ln n}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+2)^2-n^2}{n^3}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+2)^2-n^2}{n^2}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{3+2\sin(n^2)}{n^2}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)^2}{n^2}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+\ln n)^2}{n^3+n\ln n}$

More and Extra

1. Why does the integral test not directly apply to the series $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin(n)}{n^2}$? Do you think that this integral converges or diverges?
2. Using one of the formulas above, get an estimate for $\sum_{n=1}^{10{,}000} 1/n$.
3. Find $\sum_{n=1}^{5} 1/n^2$. Compute an integral to estimate the remainder $R_5 = \sum_{n=6}^{\infty} 1/n^2$.
4. Use your answer to the above problem and the fact that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ to put upper and lower bounds on π.