11.3: The Integral Test

Wednesday, February 25

Recap: Sequences

Order the following sequences by their growth rate as $n \to \infty$:

$$n, \ln(n), \sqrt{n}, 1.01^n, \sqrt{n^3+2}, n!, 50, n^2, n^{100}, 2^n, 50^n, \ln(n)^{30}, n^{1.001}$$

$$\ln(n) < \ln(n)^{30} < \sqrt{n} < n < n^{1.001} < \sqrt{n^3+2} < n^2 < n^{100} < 1.01^n < 2^n < 50^n < n!$$

Find the limits of the following sequences:

1. $\lim_{n\to\infty} \frac{n^5}{2^n} : 0$
2. $\lim_{n\to\infty} \frac{n^5}{\ln(n)^5} : \infty$
3. $\lim_{n\to\infty} \frac{e^n}{n!} : 0$
4. $\lim_{n\to\infty} \frac{n^{10}}{n!} : 0$
5. $\lim_{n\to\infty} \frac{1.01^n}{\sqrt{n}} : \infty$
6. $\lim_{n\to\infty} \frac{\ln(n)}{e^n} : 0$
7. $\lim_{n\to\infty} \frac{n^3+n^2}{\sqrt{n^7+5}} : 0$
8. $\lim_{n\to\infty} \frac{1.2^n+n^5}{1.1^n-n^3} : \infty$
9. $\lim_{n\to\infty} \frac{e^n+\sqrt{7n+2}}{\ln(n)+n!} : 0$
10. $\lim_{n\to\infty} \frac{n^4+\ln(n)}{3n^3+4n^2+n+1} : \infty$
11. $\lim_{n\to\infty} \frac{1.03^n+e^n}{n^7+2e^n} : 0$
12. $\lim_{n\to\infty} \frac{\sqrt{2n^3+5}}{3n\sqrt{n-1}+\ln(n)} : \frac{\sqrt{2}}{3}$

Recap: Geometric Series

Formula for the sum of a geometric series when $|r| < 1$:

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

Writing out the individual terms, we can get

$$a + ar + ar^2 + \ldots = a(1 + r + r^2 + \ldots) = \frac{a}{1-r}$$

So one way to find the sum is to write the first two terms of the sequence, then factor out the first term and use $1 + r + r^2 + \ldots = \frac{1}{1-r}$.

Example: $\sum_{n=1}^{\infty} \frac{3}{2^{n+2}} = 3/8 + 3/16 + \ldots = \frac{3}{8}(1 + 1/2 + \ldots) = \frac{3}{8}\frac{1}{1-1/2} = 3/4.$

1. $\sum_{n=1}^{\infty} 1/3^n = 1/3 + 1/9 + \ldots = 1/3(1 + 1/3 + \ldots) = \frac{1}{3}\frac{1}{1-1/3} = 1/2.$
2. $\sum_{n=1}^{\infty} 5/8^n = 5/8 + 5/64 + \ldots = 5/8(1 + 1/8 + \ldots) = \frac{5}{8}\frac{1}{1-1/8} = 5/7.$

3. $\displaystyle\sum_{n=0}^{\infty} 3^{n+2}/4^{n-1} = 36(1+3/4+\ldots) = 36\frac{1}{1-3/4} = 144.$

4. $\displaystyle\sum_{n=0}^{\infty} 2^{n+3}/3^{n+2} = 8/9(1+2/3+\ldots) = \frac{8}{9}\frac{1}{1-2/3} = 8/3.$

5. $\displaystyle\sum_{n=1}^{\infty} 5\cdot 2^{n}/3^{n+1} = 10/9(1+2/3+\ldots) = \frac{10}{9}\frac{1}{1-2/3} = 10/3.$

6. $\displaystyle\sum_{n=1}^{\infty} 7\cdot 2^{n+2}/5^{n} = 56/5(1+2/5+\ldots) = \frac{56}{5}\frac{1}{1-2/5} = 56/3.$

The Integral Test

1. For a positive decreasing (or eventually decreasing) sequence a_n and corresponding function f, the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.
2. $\int_1^n f(x)\,dx \le \sum_{i=1}^n a_n \le a_1 + \int_1^n f(x)\,dx$.
3. If $s = \sum a_n$ and s_n is the nth partial sum, then $\displaystyle\int_{n+1}^{\infty} f(x)\,dx \le R_n = s - s_n \le \int_n^{\infty} f(x)\,dx.$

Example: Since $a_n = 1/n$ is decreasing and $\int_1^{\infty} \frac{1}{x}\,dx$ diverges, the harmonic series diverges.
Decide whether the following series are convergent or divergent by using the integral test:

1. $\displaystyle\sum_{n=1}^{\infty} 1/n$ diverges
2. $\displaystyle\sum_{n=1}^{\infty} 1/n^2$ converges
3. $\displaystyle\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges
4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\ln(n)}$ diverges ($\int \frac{1}{x\ln x} = \ln\ln x$)
5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\ln(n)^2}$ converges ($\int \frac{1}{x\ln^2 x} = -1/\ln x$)
6. $\displaystyle\sum_{n=1}^{\infty} 1/n^3$ converges
7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges ($\int 1/(1+x^2) = \arctan(x)$)
8. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{1+n^2}$ diverges ($\int x/(1+x^2) = \ln(1+x^2)$)

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$ converges

$$\begin{aligned}
\lim_{n\to\infty} \int_1^n \frac{1}{x^2+3x+2} &= \lim_{n\to\infty} \int_1^n \frac{1}{(x+1)(x+2)} \\
&= \lim_{n\to\infty} \int_1^n \frac{1}{x+1} - \frac{1}{x+2} \\
&= \lim_{n\to\infty} \ln(x+1) - \ln(x+2)|_1^n \\
&= \ln(3) - \ln(2) + \lim_{n\to\infty} ln(n+1) - \ln(n+2) \\
&= \ln(3) - \ln(2) + \lim_{n\to\infty} \ln(\frac{n+1}{n+2}) \\
&= \ln(3) - \ln(2) + \ln(\lim_{n\to\infty} 1 + \frac{1}{n+2}) \\
&= \ln(3) - \ln(2)
\end{aligned}$$

Decide whether the followng integrals are convergent or divergent by using the integral test. You do not have to compute the integral.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+\sqrt{n}}{n^3+\ln n}$ diverges (like $1/n$)

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^6-n^5}{n^4+n^3+\sin(n)}$ diverges (like n^2)

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^3}{n^5+7}$ converges (like $1/n^2$)

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^3-n^2+n}{n^2+\ln n}$ diverges (like n)

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+2)^2-n^2}{n^3}$ converges (like $4/n^2$). Careful with this one–the higher order terms in the numerator cancel out!

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+2)^2-n^2}{n^2}$ diverges (like $1/n$)

7. $\displaystyle\sum_{n=1}^{\infty} \frac{3+2\sin(n^2)}{n^2}$ converges (like $3/n^2$)

8. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)^2}{n^2}$ converges (we will cover this more on Friday, but since $\ln^2 n < \sqrt{n}$ for large n, the series can be compared to $\sqrt{n}/n^2 = 1/n^{3/2}$)

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+\ln n)^2}{n^3+n\ln n}$ diverges (like $1/n$)

More and Extra

1. Why does the integral test not directly apply to the series $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin(n)}{n^2}$? Do you think that this integral converges or diverges?

   Due to the oscillation of $\sin(n)$ the sequence is not decreasing. The integral converges.

2. Using one of the formulas above, get an estimate for $\sum_{n=1}^{10,000} 1/n$.

$$\int_1^{10,000} 1/x\,dx \leq \sum_{n=1}^{10,000} 1/n \leq 1 + \int_1^{10,000} 1/x\,dx$$

$$\ln(10,000) \leq \sum_{n=1}^{10,000} 1/n \leq 1 + \ln(10,000)$$

$$9.21 \leq \sum_{n=1}^{10,000} 1/n \leq 10.21$$

3. Find $\sum_{n=1}^{5} 1/n^2$. Compute an integral to estimate the remainder $R_5 = \sum_{n=6}^{\infty} 1/n^2$.

$$\int_{n+1}^{\infty} 1/x^2\,dx \leq R_5 \leq \int_n^{\infty} 1/x^2\,dx$$

$$1/(n+1) \leq R_n \leq 1/n$$

   $\sum_{n=1}^{5} 1/n^2 = 5269/3600 \approx 1.4636$. From the above derivation for R_n, we get $1/6 \leq R_n \leq 1/5$

4. Use your answer to the above problem and the fact that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ to put upper and lower bounds on π.

$$1/6 \leq \pi^2/6 - s_5 \leq 1/5$$

$$s_5 + 1/6 \leq \pi^2/6 \leq s_5 + 1/5$$

$$1.630 \leq \pi^2/6 \leq 1.6636$$

$$3.127 \leq \pi \leq 3.1594$$