# Integration by Parts–Solutions

Wednesday, January 21

## Tips

- "L-I-A-T-E"

  When in doubt, a good heuristic is to choose $u$ to be the first type of function in the following list:

  1. L-Logartithmic functions ($\ln(x), \log_2(x)$).
  2. I-Inverse trig functions ($\arcsin(x), \arccos(x), \arctan(x)$.
  3. A-Algebraic functions ($x^2 + 2x, 1/x, \sqrt{1+x^2}$).
  4. T-Trig functions ($\sin(x), \cos(x), \tan(x)$).
  5. E-Exponential functions ($e^{2x}, 5^x$).

- If that doesn't lead to a clear choice for $u$, instead choose $dv$ so that it can be easily integrated.

  **Example:** If we want to evaluate $\int x^3/\sqrt{1+x^2}\,dx$, there is not a clear choice for $u$ since the entire function is algebraic. If we look at choices for $dv$ instead, we can see that although $1/\sqrt{1+x^2}$ might not have a simple antiderivative, $x/\sqrt{1+x^2}$ has $\sqrt{1+x^2}$ as an antiderivative. So we should choose $dv = x/\sqrt{1+x^2}$ and $u = x^2$.

## Warm-up

Use integration by parts to evaluate the following integrals:

1. $\int xe^x\,dx$

   Pick $u = x, dv = e^x\,dx$. Then $\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x$.

2. $\int_1^e \ln(x)\,dx$

   Pick $u = \ln(x), dv = dx$. Then $\int_1^e \ln(x)\,dx = x\ln(x)|_1^e - \int_1^e dx = e - e + 1 = 1$.

3. $\int x^3\sqrt{1+x^2}\,dx$

   Let $u = x^2, dv = x\sqrt{1+x^2}\,dx$, so $v = \frac{1}{3}(1+x^2)^{3/2}, du = 2x\,dx$. Then $\int x^3\sqrt{1+x^2}\,dx = \frac{x^2}{3}(1+x^2)^{3/2} - \frac{1}{3}\int 2x(1+x^2)^{3/2} = \frac{x^2}{3}(1+x^2)^{3/2} - \frac{2}{15}(1+x^2)^{5/2}$.

   Alternatively, substitute $y = x^2$ to get $\frac{1}{2}\int y\sqrt{1+y}\,dy$ to get a slightly simpler problem. Then let $u = y, dv = \sqrt{1+y}\,dy$ and get $\frac{1}{3}y(1+y)^{3/2} - \frac{1}{3}\int(1+y)^{3/2} = \frac{1}{3}y(1+y)^{3/2} - \frac{2}{15}(1+y)^{5/2} = \frac{1}{3}x^2(1+x^2)^{3/2} - \frac{2}{15}(1+x^2)^{5/2}$.

## Speed Round

For the following problems, you do not have to evaluate the integral. Simply circle the part of the function that you would set equal to $u$ ($dv$ will implicitly be everything you did not circle). Answer each batch of problems as quickly and as accurately as you can.

### Round 1

1. $\int xe^{3x}\,dx$: u = x
2. $\int x\sin(5x)\,dx$: u = x
3. $\int x^2\ln(x)\,dx$: u = $\ln x$

4. $\int x^2 e^{-5x}\,dx$: $u = x^2$
5. $\int x^2 \cos(3x)\,dx$: $u = x^2$
6. $\int (x^3+4)e^{2x}\,dx$: $u = x^3 + 4$ (or split the integral into two parts first)
7. $\int \arcsin(x)\,dx$: $u = \arcsin(x)$
8. $\int x \arctan(x)\,dx$: $u = \arctan(x)$
9. $\int x \ln(x)\,dx$: $u = \ln(x)$

**Round 2**

This time the LIATE rule might not always get you the answer. Write down your choice of $u$ if you cannot circle it properly.

1. $\int x^2 2^x\,dx$: u $= x^2$
2. $\int \ln(x)/\sqrt{x}\,dx$: $u = \ln(x)$
3. $\int x^3/\sqrt{1+x^2}\,dx$: $u = x^2$ (or substitute $y = x^2$ first)
4. $\int e^x \sin(2x)\,dx$: Either choice is okay; you will have to integrate parts twice before getting the integral you started with as part of a larger expression.
5. $\int x^5 e^{x^2}\,dx$: $u = x^4$ (or substitute $y = x^2$ first)
6. $\int \ln(x)/x^2\,dx$: $u = \ln x$
7. $\int e^{2x} \sin(e^x)\,dx$: $u = e^x$ (or substitute $y = e^x$ first)
8. $\int x \ln^2(x)\,dx$: $u = \ln^2(x)$
9. $\int \arctan(x)\,dx$: $u = \arctan(x)$

**Round 3**

Using integration by parts might not always be the correct (or best) solution. For the following problems, indicate whether you would use integration by parts (with your choices of $u$ and $dv$), substitution (with your choice of $u$), or neither.

1. $\int x \ln(x)\,dx$: IBP: $u = \ln x$
2. $\int \ln(x)/x\,dx$: sub $y = \ln x$
3. $\int e^x\,dx$: just integrate
4. $\int \frac{1}{x\ln(x)}\,dx$: sub $y = \ln x$
5. $\int \arccos(2x)\,dx$: IBP $u = \arccos(2x)$
6. $\int e^{\cos(x)} \sin(x)\,dx$: sub $y = \cos x$
7. $\int e^x \sin(x)\,dx$: IBP $u = e^x$ (or $u = \sin x$)
8. $\int x^3 \cos(x^2)\,dx$: IBP $u = x^2$ (or sub $y = x^2$)
9. $\int 1/(1+x^2)\,dx$: just integrate (it's arctan)
10. $\int x \tan^2(x)\,dx$: IBP $u = x$
11. $\int x^5 e^{-x^2}\,dx$: IBP $u = x^4$ (or sub $y = x^2$)
12. $\int x + \sin(x)\,dx$: just integrate.

## Bonus

Evaluate $\int_0^1 x^5 e^{-x}$ using integration by parts. Then use the fact that $0 \leq \int_0^1 x^5 e^{-x} \leq 1$ (why?) to put an upper and lower bound on $e$.
Integrating by parts five times in a row gives

$$\int_0^1 x^5 e^{-x} = -e^{-x}(x^5 + 5x^4 + 20x^3 + 60x^2 + 120x + 120)|_0^1 = 120 - 326/e$$

The integral must be greater than zero because the function is positive on [0,1], and it must be less than 1 since the function is less than 1 on [0,1]. Therefore

$$\begin{aligned} 0 &\leq 120 - 326/e \\ 0 &\leq 120e - 326 \\ 326/120 &\leq e \end{aligned}$$

$$\begin{aligned} 1 &\geq 120 - 326/e \\ e &\geq 120e - 326 \\ 326/119 &\geq e \end{aligned}$$

Put together, this gives the (approximate) bounds $2.71 \leq e \leq 2.74$. Evaluating an integral with a higher power of $x$ will give an even more accurate bound.