Improper Integrals: Solutions

Friday, February 6

Bounding Functions

Put guaranteed upper and lower bounds on the following, with justification:
These answers put bounds on the absolute values of the functions, which automatically provide upper and lower bounds since $|x| \leq K \Leftrightarrow -K \leq x \leq K$

1. $\sin(x) + x\cos(x), x \in [2, 5]$

   $|\sin(x) + x\cos(x)| \leq |\sin(x)| + |x| \cdot |\cos(x)| \leq 6$

2. $x^2 + x, x \in [-1, 1]$

   $|x^2 + x| \leq x^2 + |x| \leq 2$

3. $\sqrt{\ln(x) + x}, x \in [4, 12]$

   $|\sqrt{\ln(x) + x}| \leq \sqrt{\ln(12) + 12} \leq \sqrt{12 + 12} \leq 5$

4. $e^{1/x} + x - x^2, x \in [2, 4]$

   $|e^{1/x} + x - x^2| \leq e^{1/x} + x + x^2 \leq \sqrt{e} + 20 \leq 22$

5. $\sin(\sin(\sin(x^5))), x \in [-10, 10]$

   $|\sin(\cdots)| \leq 1$

6. $2x\sin(x) - 4x^2\cos(x) + e^{x^2}, x \in [0, 1]$

   $|2x\sin(x) - 4x^2\cos(x) + e^{x^2}| \leq 2 + 4 + e \leq 9$

7. The total distance a human can run in a day.

   The world record for a marathon is a little over two hours, so it's safe to say that no human can run more than 30 miles in two hours. Multiplying by 12 gives a cap of 240 miles in a single day.

   In 2005, Dean Karnazes ran 350 miles non-stop in 80 hours and 44 minutes, which works out to an average of a little over 104 miles per day.

8. The combined weight of all people living in California.

   California's population is less than 100 million, and everybody in the state weighs less than 2000 pounds, so a total of 200 billion pounds should be a safe cap.

Infinite Intervals

Determine whether each of the following integrals are convergent or divergent:

1. $\int_1^\infty \frac{1}{x}\,dx$ divergent (p-test)

2. $\int_1^\infty e^{-x}\,dx$ convergent (integrate)

3. $\int_1^\infty \frac{1}{x^2}\,dx$ convergent (p-test)

4. $\int_1^\infty \sin(x)\,dx$ divergent (oscillates)

5. $\int_1^\infty xe^{-x}\,dx$ convergent (integrate by parts)

6. $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ (p-test)

7. $\int_1^\infty e^{-x^2}\,dx$ convergent (compare to e^{-x})

8. $\int_1^\infty \frac{1}{x^{1.003}}\,dx$ convergent (p-test)

9. $\int_1^\infty \frac{100000}{x^{1.003}}\,dx$ convergent (p-test)

10. $\int_1^\infty \frac{1}{\ln(x)}\,dx$ divergent (compare to 1/x, or integrate by parts)

11. $\int_1^\infty \frac{1}{x\ln(x)}\,dx$ divergent (sub $u = \ln(x)$)

12. $\int_1^\infty \frac{1}{x\ln^2(x)}\,dx$ convergent (sub $u = \ln(x)$)

Basic Rules for Convergence:

1. If $\int f$ is convergent, then $\int Kf$ is convergent for any constant K.
2. If $\int f$ and $\int g$ are convergent, then $\int (f + g)$ is convergent.
3. If $\int f$ is convergent and $0 \leq g(x) \leq f(x)$ for all x, then $\int g$ is convergent.

Comparison Test

Find an appropriate "model function" to compare each of the following functions to. Decide whether each integral is convergent or divergent.

1. $\int_1^\infty \frac{1}{(2x+1)^3}\,dx$ – $1/x^3$

2. $\int_1^\infty \frac{1}{2x^2 - x}\,dx$ – $1/x^2$

3. $\int_1^\infty \frac{x^2}{3 + x^3}\,dx$ – $1/x$

4. $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ – $1/x^2$

5. $\int_1^\infty \frac{1}{x + 3\ln(x)}\,dx$ – $1/x$

6. $\int_1^\infty \frac{1}{x^2 - 3x + 2}\,dx$ – $1/x^2$

7. $\int_1^\infty \frac{3 + \sin(x)}{e^x}\,dx$ – e^{-x} (The original problem was $(3 + \sin(x))/e^{-x}$, which was divergent)

8. $\int_1^\infty \frac{1+\ln(x)}{x}\,dx$ – $1/x$

9. $\int_1^\infty \frac{x^2+3x+2}{x^4-x+1}\,dx$ – $1/x^2$

Singularities

Decide whether each of the following integrals are convergent or divergent:

1. $\int_0^1 \frac{1}{x}\,dx$ divergent (p-test)

2. $\int_0^1 \frac{1}{x^2}\,dx$ divergent (p-test)

3. $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ convergent (p-test)

4. $\int_2^4 \frac{1}{x-3}\,dx$ divergent (p-test at $x=3$)

5. $\int_4^6 \frac{1}{x-3}\,dx$ convergent (no singularity)

6. $\int_1^2 \frac{1}{x^2-1}\,dx$ divergent (p-test at $x=1$)

7. $\int_0^1 \frac{100000}{\sqrt{x}}\,dx$ convergent (p-test)

8. $\int_5^6 \frac{1}{(x-3)\sqrt{x-5}}\,dx$ convergent (p-test) [The original problem had a lower limit of 4, which would have made $\sqrt{x-5}$ undefined on part of the interval.]

9. $\int_0^1 x^2\,dx$ divergent (p-test, again)

More Comparison Test

Pretty much the only function you care to compare things to here is $1/x$ (or, more generally, $1/x^p$).
Here, the answer just provides the base function that the problem most closely resembles. The function being tested is not necessarily larger or smaller than that base function, but it will be within some constant factor of that function as $x \to 0$.

1. $\int_0^3 \frac{x}{x-2}\,dx$ –divergent ($1/(x-2)$ at $x=2$)

2. $\int_6^1 0\frac{(x-4)(3x+1)}{\sqrt{x-6}}$ – convergent ($1/\sqrt{x-6}$ at $x=6$) [The original integral had bounds of 4 and 6, which would make $\sqrt{x-6}$ undefined.]

3. $\int_{-1}^1 \frac{3x+5}{x^2+2x+1}$– divergent ($1/(x+1)^2$ at $x=-1$)

4. $\int_0^1 \frac{x^2 - x + 3}{x^2 - x}$– divergent ($1/(x-1)$ at $x = 1$ and/or $1/x$ at $x = 0$)

5. $\int_1^2 \frac{3x}{x-1}$– divergent ($1/(x-1)$)

6. $\int_4^5 \frac{x + \ln(x) + 5\sqrt{x}}{\sqrt{x-4}}$– [The original integral had bounds of 0 and 4, which would make $\sqrt{x-4}$ undefined.]

7. $\int_0^1 \frac{3x+1}{e^x}$– convergent (no singularity)

8. $\int_0^2 \frac{\sqrt{x} + 2e^{-x} - e^x}{x^{2/3}}$– convergent (compare to $1/x^{2/3}$ at $x = 0$)[The original integral had a lower bound of -2, which would make $\sqrt{x}$ undefined.]

Harder Limits

1. $\int_0^1 \frac{\sin(x)}{x}\,dx$ – convergent since $\lim_{x\to 0} \sin(x)/x = 1$.

2. $\int_0^1 x\ln(x)\,dx$ – convergent since $\lim_{x\to 0} x\ln x = 0$, so the function is bounded.

3. $\int_{-1}^0 \frac{e^{1/x}}{x^3}\,dx$– convergent since $\lim_{x\to 0^-} e^{1/x}/x^3 = \lim_{x\to -\infty} x^3 e^x = 0$ (by L'Hospital's Rule)

4. $\int_0^1 \frac{e^{1/x}}{x^3}\,dx$– very divergent ($e^{1/x} > 1$, so just compare to $1/x^3$)

5. $\int_0^1 \frac{\ln(1+x)}{x}\,dx$– Convergent since $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$, so the function is bounded.

6. $\int_0^1 \frac{\arctan x}{x}\,dx$– convergent since the limit as $x \to 0$ is 1 (by L'Hospital), so the function is bounded.