11.6: Convergence Tests

Friday, March 6

Speed Round

1. $\cos(\pi n) = (-1)^n$
2. $\frac{2^{n+1}}{2^n} = 2$
3. $\frac{(n+1)!}{n!} = n+1$
4. $\frac{(2n+2)!}{2n!} = (2n+1)(2n+2)$
5. $\frac{2^{n+1}}{n!}\frac{(n+1)!}{2^n} = 2(n+1)$
6. $\lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 1$
7. $\lim_{n\to\infty} n^{1/n} = 1$
8. $\lim_{n\to\infty} e^{1/n} = 1$
9. $\lim_{n\to\infty} e^{-1/n} = 1$

Convergent or divergent?

1. $\sum_{n=1}^{\infty} \frac{1}{n + \ln n}$ divergent
2. $\sum_{n=1}^{\infty} \frac{n^2+3n}{n^3}$ divergent
3. $\sum_{n=1}^{\infty} \frac{n+\sqrt{n^3+3}}{n^3}$ convergent
4. $\sum_{n=1}^{\infty} \frac{n^2 2^n}{n!}$ convergent
5. $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ convergent
6. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ convergent
7. $\sum_{n=1}^{\infty} \frac{n^3+\pi^n+\sin n}{3^n}$ divergent
8. $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+\sqrt{2}}$ convergent
9. $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$ divergent

The Convergence Tests

1. Alternating Series: If
   (a) $b_n > 0$ for all n
   (b) b_n is a monotonically decreasing sequence ($b_n > b_{n+1}$)
   (c) $\lim_{n\to\infty} b_n = 0$,

   then $\sum_{n=1}^{\infty}(-1)^n b_n$ converges.
2. Ratio Test: If $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges. If $L > 1$, the series diverges, and if $L = 1$ the test is inconclusive.
3. Root test: If $L = \lim_{n\to\infty} |a_n|^{1/n} < 1$ then $\sum_{n=1}^{\infty} a_n$ converges. If $L > 1$, the series diverges, and if $L = 1$ the test is inconclusive.

The Ratio Test and Root Test Do Not Care For Polynomials

Let $a_n = n^K$, for any number K. Show that the ratio and root tests are both inconclusive.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^K}{n^K} = \lim_{n\to\infty} (1+1/n)^K = 1$$

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} n^{K/n} = \lim_{n\to\infty} e^{K\ln n/n} = e^{K \lim_{n\to\infty} \ln n/n} = e^0 = 1$$

Exercises

Decide whether the following series are absolutely convergent, conditionally convergent, or divergent.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ CC
2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ ABS
3. $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ ABS
4. $\sum_{n=1}^{\infty} \frac{n^2 2^n}{3^n}$ ABS
5. $\sum_{n=1}^{\infty} \frac{n^{10} 5^n}{n!}$ ABS
6. $\sum_{n=1}^{\infty} \frac{n^3 + (-5)^n}{4^n}$ DIV
7. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n - \ln n}$ CC
8. $\sum_{n=1}^{\infty} \frac{n + (-3)^n}{3^n}$ DIV
9. $\sum_{n=1}^{\infty} \left(\frac{n^3 + 4\sqrt{n}}{2n^3 + 1}\right)^n$ ABS
10. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ ABS
11. $\sum_{n=1}^{\infty} \frac{(-1)^n (n + 2^n)}{n 2^n}$ CC
12. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ ABS

Our Favorite Power Series

For each of the following power series, find the values of x for which the series is divergent, the values for which it is absolutely convergent, and the values (if any) for which it is conditionally convergent.

1. $e^x = 1 + x + x^2/2 + x^3/3! + \ldots = \sum_{n=1}^{\infty} \frac{x^n}{n!}$

   Absolutely convergent for all x (use the Ratio Test).

2. $\sin(x) = x - x^3/3! + x^5/5! - \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$

   Absolutely convergent for all x (use the Ratio Test).

3. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$

   Absolutely convergent for $|x| < 1$, divergent for $|x| \geq 1$ (Ratio Test).

4. $\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

   Absolutely convergent for $|x| < 1$, conditionally convergent for $x = 1$, divergent otherwise.

5. $\arctan(x) = x - x^3/3 + x^5/5 - \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1}$

   Absolutely convergent for $|x| < 1$, conditionally convergent for $x = \pm 1$, divergent otherwise.