11.5-11.6: Review

Monday, March 9

Speed Round

Evaluate the expressions, or determine whether the series converge or diverge:

1. $\cos(\pi n) = (-1)^n$
2. $e^x = 1 + x + x^2/2! + x^3/3! + \dots$
3. $\sin(x) = x - x^3/3! + x^5/5! - \dots$
4. $\frac{(n+2)!}{n!} = (n+1)(n+2)$
5. $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$
6. $\frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n} = \frac{3}{n+1}$
7. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges
8. $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ converges
9. $\sum_{n=1}^{\infty}\frac{(\ln n)^3}{n^2}$ converges
10. $\sum_{n=1}^{\infty}\frac{1+\sqrt{n}}{n^2}$ converges
11. $\sum_{n=1}^{\infty}\frac{1+\sqrt{n}}{n\sqrt{n}}$ diverges
12. $\sum_{n=1}^{\infty}\frac{2^n\cdot n^{100}}{n!}$ converges
13. $\sum_{n=1}^{\infty}\frac{(-\pi)^n}{3^n}$ diverges
14. $\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$ converges
15. $\sum_{n=1}^{\infty}\frac{n^2+2^n}{3^n}$ converges

Alternating Series

Draw a picture illustrating the proof of the Alternating Series test:
See page 728 of Stewart. The main points are that the series 1) is alternating, 2) has terms converging to zero, and 3) has terms that are monotonically decreasing in magnitude.

Which of the following series does the Alternating Series test apply to?

1. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ yes
2. $\sum_{n=1}^{\infty}\frac{1}{n}$ no: not alternating
3. $\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\ln n}$ yes
4. $\sum_{n=1}^{\infty}\frac{\cos n}{n}$ no: not alternating
5. $\sum_{n=1}^{\infty}\frac{(-1)^n\ln n}{1+\ln n}$ no: $\lim_{n\to\infty} a_n \neq 0$

6. $\sum_{n=1}^{\infty} \frac{1+2(-1)^n}{n}$ no: terms are not decreasing in magnitude.

Absolute Convergence

Find three different series that are convergent but not absolutely convergent. What do they have in common? $\frac{(-1)^n}{n}, \frac{(-1)^n}{\sqrt{n}}, \frac{(-1)^n}{\ln n}, \frac{(-1)^n}{n \ln n}$ all converge to zero but very slowly (in particular, they must decay more slowly than $|a_n| = \frac{1}{n^p}$ for any $p > 1$).

Various Tests

Decide whether the following series are absolutely convergent, conditionally convergent, or divergent. Decide which test is most appropriate.

1. $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ ABS
2. $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ ABS (ratio test)
3. $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ ABS (ratio test)
4. $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)(n!)}$ DIV (ratio test)
5. $\sum_{n=1}^{\infty} \frac{n^2 \cdot 3^n}{n!}$ ABS (ratio test)
6. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ CC (alternating series test)
7. $\sum_{n=1}^{\infty} \frac{3+(-1)^n}{n^{3/2}}$ ABS (limit comparison test)
8. $\sum_{n=1}^{\infty} \left(\frac{1+2\ln n}{3\ln n}\right)^n$ ABS (root test)
9. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sin n}$ DIV ($\lim_{n\to\infty} a_n \neq 0$)
10. $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{1+\ln n}$ CC (alternating series test)
11. $\sum_{n=1}^{\infty} \frac{2^n \cdot n!}{(n+3)!}$ DIV (ratio test)
12. $\sum_{n=1}^{\infty} (-1)^n \sin(1/n)$ CC (alternating series test) [to show it is not absolutely convergent, use Limit Comparison with $b_n = 1/n$]

Some Assembly Required

Decide whether the following series are absolutely convergent, conditionally convergent, or divergent.

1. $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n^2}$ ABS: use the root test to get $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = 1/e$.
2. $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right)$ CC: multiply by the conjugate radical.
3. $\sum_{n=1}^{\infty} (-1)^n \sqrt{n}\, (\ln(n+1) - \ln n)$ CC: Use $\ln(n+1) - \ln n = \ln(1 + 1/n) \sim 1/n$ and alternating series test.
4. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ DIV (ratio test, or just $\lim_{n\to\infty} a_n = \infty$)
5. $\sum_{n=1}^{\infty} (-1)^n n \tan(\pi/n)$ DIV ($\lim_{n\to\infty} a_n \neq 0$, using $\tan(\pi/n) \sim \pi/n$ as $n \to \infty$)
6. $\sum_{n=1}^{\infty} \sin \frac{\pi}{n^2} - \sin \frac{\pi}{n^3}$ ABS (use $\sin x \sim x$ as $x \to 0$)

Power Series

For what values of x will $\sum_{n=1}^{\infty} x^n/n$ converge?
Absolutely convergent for $|x| < 1$, conditionally convergent for $x = -1$, divergent otherwise.