

# 11.4: The Comparison Test

Wednesday, February 25

## Speed Round

1.  $\lim_{n \rightarrow \infty} \ln(n)^3/n$

5.  $\lim_{n \rightarrow \infty} \frac{n^{100} e^{4n}}{n!}$

10.  $\sum_{n=0}^{\infty} 1/3^n$

2.  $\lim_{n \rightarrow \infty} 1.04^n/n^2$

6.  $\lim_{n \rightarrow \infty} \frac{n^2 2^n}{3^n}$

11.  $\sum_{n=0}^{\infty} 5/4^n$

3.  $\lim_{n \rightarrow \infty} e^n/\ln(n)$

7.  $e^x = 1 + x + \dots$

12.  $\sum_{n=1}^{\infty} 3 \cdot 2^{n+1}/5^n$

4.  $\lim_{n \rightarrow \infty} \sin(n^2)/\ln(n)$

8.  $\sin(x) = x - \dots$

9.  $\cos(x) = 1 - \dots$

Determine whether the following series are convergent or divergent:

1.  $\sum_{n=1}^{\infty} 0.8^n$

4.  $\sum_{n=1}^{\infty} \sin(n)$

7.  $\sum_{n=1}^{\infty} \frac{1}{n^2}$

2.  $\sum_{n=1}^{\infty} (-1/2)^n$

5.  $\sum_{n=1}^{\infty} \frac{n+1}{n+2}$

8.  $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

3.  $\sum_{n=1}^{\infty} (-1.3)^n$

6.  $\sum_{n=1}^{\infty} \frac{1}{n}$

9.  $\sum_{n=1}^{\infty} \frac{1}{n^{1.001}}$

## The Limit Comparison Test

- Suppose that  $\sum a_n, \sum b_n$  are series with positive terms. If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = C$  for some  $0 < C < \infty$ , then  $\sum a_n$  and  $\sum b_n$  either both converge or both diverge.
- If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$  and  $\sum b_n$  converges, then  $\sum a_n$  converges.
- If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$  and  $\sum b_n$  diverges, then  $\sum a_n$  diverges.

Example: If  $a_n = \frac{n^2 + \ln(n)^3}{n^4 + \sin(n)}$ , then let  $b_n = \frac{n^2}{n^4} = \frac{1}{n^2}$ . Then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{(n^2 + \ln(n)^3)n^2}{n^4 + \sin(n)} = \lim_{n \rightarrow \infty} \frac{1 + \ln(n)^3/n^2}{1 + \sin(n)/n^2} = 1,$$

so  $\sum a_n$  converges because  $\sum b_n$  converges.

1.  $\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$

4.  $\sum_{n=1}^{\infty} \frac{3^n + n^3}{n^4 + \ln n + 4^n}$

7.  $\sum_{n=1}^{\infty} \frac{1}{1 + \ln n + n^2}$

2.  $\sum_{n=1}^{\infty} \frac{n^3}{n^4 + 3}$

5.  $\sum_{n=1}^{\infty} \frac{n^2 + 1}{\sqrt{3 + n^7}}$

8.  $\sum_{n=1}^{\infty} \frac{n^2 + \sqrt{n^2 + 1}}{n^3 + \sqrt{n^3 + 1}}$

3.  $\sum_{n=1}^{\infty} \frac{n + \sin n}{n^3 + \ln n}$

6.  $\sum_{n=1}^{\infty} \frac{2^n + \ln n}{n + 3^n}$

9.  $\sum_{n=1}^{\infty} \frac{e^n - n}{2^n + n}$

## Different Growth Levels in the Numerator and Denominator

Good news: The dominant term is still the only thing that matters! Here's why, in a nutshell:

1.  $\ln(n)^K < n^\epsilon$  (for large  $n$ , given any  $K, \epsilon > 0$ ).
2.  $n^K < (1 + \epsilon)^n$  (for large  $n$ , given any  $K, \epsilon > 0$ ).

**Example:** Looking at  $\sum \frac{\ln^5(n)}{n^2}$ , we know that  $\ln^5(n) < \sqrt{n}$  for large values of  $n$  (in particular, when  $n > 3.44 \times 10^{15}$ ), so we can choose  $b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$ . Then

$$\lim_{n \rightarrow \infty} a_n/b_n = \lim_{n \rightarrow \infty} \frac{\ln^5(n)n^{3/2}}{n^2} = \lim_{n \rightarrow \infty} \frac{\ln^5(n)}{\sqrt{n}} = 0.$$

Since  $\sum 1/n^{3/2}$  converges, so does  $\sum \frac{\ln^5(n)}{n^2}$ .

**Example 2:** Looking at  $\sum n^2/2^n$ , we know that  $n^2 < 1.5^n$  for large values of  $n$  (when  $n \geq 13$ ), so choose  $b_n = 1.5^n/2^n$ . Then  $\lim_{n \rightarrow \infty} a_n/b_n = \lim_{n \rightarrow \infty} n^2/1.5^n = 0$ . Since  $\sum 1.5^n/2^n$  converges, so does  $\sum n^2/2^n$ .

Find an appropriate series  $b_n$  to compare each of the following series to, and determine whether they converge or diverge:

1.  $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

3.  $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

5.  $\sum_{n=1}^{\infty} \frac{n^2 2^n}{1 + 3^n}$

2.  $\sum_{n=1}^{\infty} \frac{\ln n + \sin n}{n^2}$

4.  $\sum_{n=1}^{\infty} \frac{n}{1.02^n}$

6.  $\sum_{n=1}^{\infty} \frac{n \ln n}{\sqrt{n^3 + 2}}$

### Warning!

If terms in the numerator or denominator are cancelling out with each other, watch out! You might have to do some extra algebra before using the comparison test. Try to show that the following series converge:

1.  $\sum_{n=1}^{\infty} \frac{\sqrt{n^2 + 1} - n}{n}$

3.  $\sum_{n=1}^{\infty} \frac{e^{1/n} - 1}{n}$

2.  $\sum_{n=1}^{\infty} \frac{\ln(n+1) - \ln(n)}{n}$

4.  $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n + \pi}$

### Bonus

1. Show that  $1 + 1/1! + 1/2! + 1/3! + \dots$  converges. (Hint:  $3 \cdot 2 > 2 \cdot 2$ ,  $4 \cdot 3 \cdot 2 > 2 \cdot 2 \cdot 2 \dots$ )
2. For what values of  $x$  does the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  converge?
3. If we integrate both sides of  $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ , we get  $-\ln(1-x) = x + x^2/2 + x^3/3 + \dots$  (this actually works!) For what values of  $x$  does the series representing  $-\ln(1-x)$  converge?