11.4: The Comparison Test

Wednesday, February 25

Speed Round

1. $\lim_{n\to\infty} \ln(n)^3/n = 0$
2. $\lim_{n\to\infty} 1.04^n/n^2 = \infty$
3. $\lim_{n\to\infty} e^n/\ln(n) = \infty$
4. $\lim_{n\to\infty} \sin(n^2)/\ln(n) = 0$
5. $\lim_{n\to\infty} \frac{n^{100}e^{4n}}{n!} = 0$
6. $\lim_{n\to\infty} \frac{n^2 2^n}{3^n} = 0$
7. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$
8. $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$
9. $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$
10. $\sum_{n=0}^{\infty} 1/3^n = \frac{1}{1-1/3} = 3/2$
11. $\sum_{n=0}^{\infty} 5/4^n = 5\left(\frac{1}{1-1/4}\right) = 20/3$
12. $\sum_{n=1}^{\infty} 3 \cdot 2^{n+1}/5^n = \frac{12}{5}(1 + 2/5 + \ldots) = \frac{12}{5}\left(\frac{1}{1-2/5}\right) = 4$

Determine whether the following series are convergent or divergent:

1. $\sum_{n=1}^{\infty} 0.8^n$ converges
2. $\sum_{n=1}^{\infty} (-1/2)^n$ converges
3. $\sum_{n=1}^{\infty} (-1.3)^n$ diverges
4. $\sum_{n=1}^{\infty} \sin(n)$ diverges
5. $\sum_{n=1}^{\infty} \frac{n+1}{n+2}$ diverges ($\lim_{n\to\infty} a_n \neq 0$)
6. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges
7. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges
8. $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges
9. $\sum_{n=1}^{\infty} \frac{1}{n^{1.001}}$ converges

The Limit Comparison Test

1. Suppose that $\sum a_n, \sum b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = C$ for some $0 < C < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Example: If $a_n = \frac{n^2+\ln(n)^3}{n^4+\sin(n)}$, then let $b_n = \frac{n^2}{n^4} = \frac{1}{n^2}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n^2+\ln(n)^3)n^2}{n^4+\sin(n)} = \lim_{n\to\infty} \frac{1+\ln(n)^3/n^2}{1+\sin(n)/n^2} = 1,$$

so $\sum a_n$ converges because $\sum b_n$ converges.

1. $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges ($b_n = 1/2^n$)
2. $\sum_{n=1}^{\infty} \frac{n^3}{n^4+3}$ diverges ($b_n = 1/n$)
3. $\sum_{n=1}^{\infty} \frac{n+\sin n}{n^3+\ln n}$ converges ($b_n = 1/n^2$)
4. $\sum_{n=1}^{\infty} \frac{3^n+n^3}{n^4+\ln n+4^n}$ converges ($b_n = (3/4)^n$)
5. $\sum_{n=1}^{\infty} \frac{n^2+1}{\sqrt{3+n^7}}$ converges ($b_n = 1/n^{3/2}$)
6. $\sum_{n=1}^{\infty} \frac{2^n+\ln n}{n+3^n}$ converges ($b_n = (2/3)^n$)
7. $\sum_{n=1}^{\infty} \frac{1}{1+\ln n+n^2}$ converges ($b_n = 1/n^2$)
8. $\sum_{n=1}^{\infty} \frac{n^2+\sqrt{n^2+1}}{n^3+\sqrt{n^3+1}}$ diverges ($b_n = 1/n$)
9. $\sum_{n=1}^{\infty} \frac{e^n-n}{2^n+n}$ diverges ($b_n = (e/2)^n$)

Different Growth Levels in the Numerator and Denominator

Good news: The dominant term is still the only thing that matters! Here's why, in a nutshell:

1. $\ln(n)^K < n^\epsilon$ (for large n, given any $K, \epsilon > 0$).
2. $n^K < (1+\epsilon)^n$ (for large n, given any $K, \epsilon > 0$).

Example: Looking at $\sum \frac{\ln^5(n)}{n^2}$, we know that $\ln^5(n) < \sqrt{n}$ for large values of n (in particular, when $n > 3.44 \times 10^{15}$), so we can choose $b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$. Then

$$\lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} \frac{\ln^5(n) n^{3/2}}{n^2} = \lim_{n\to\infty} \frac{\ln^5(n)}{\sqrt{n}} = 0.$$

Since $\sum 1/n^{3/2}$ converges, so does $\sum \frac{\ln^5(n)}{n^2}$.
Example 2: Looking at $\sum n^2/2^n$, we know that $n^2 < 1.5^n$ for large values of n (when $n \geq 13$), so choose $b_n = 1.5^n/2^n$. Then $\lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} n^2/1.5^n = 0$. Since $\sum 1.5^n/2^n$ converges, so does $\sum n^2/2^n$.

Find an appropriate series b_n to compare each of the following series to, and determine whether they converge or diverge:

1. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges ($b_n = 1/n$)
2. $\sum_{n=1}^{\infty} \frac{\ln n+\sin n}{n^2}$ converges ($b_n = \sqrt{n}/n^2$)
3. $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges ($b_n = 2^n/3^n$)
4. $\sum_{n=1}^{\infty} \frac{n}{1.02^n}$ converges ($b_n = 1.01^n/1.02^n$)
5. $\sum_{n=1}^{\infty} \frac{n^2 2^n}{1+3^n}$ converges ($b_n = 2.1^n/3^n$)
6. $\sum_{n=1}^{\infty} \frac{n\ln n}{\sqrt{n^3+2}}$ diverges ($b_n = 1/\sqrt{n}$)

Warning!

If terms in the numerator or denominator are cancelling out with each other, watch out! You might have to do some extra algebra before using the comparison test. Try to show that the following series converge:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}-n}{n}$ converges ($b_n = 1/n^2$)

$$\begin{aligned}\lim_{n\to\infty} \frac{\sqrt{n^2+1}-n}{n}(n^2) &= \lim_{n\to\infty} \frac{\sqrt{n^2+1}-n}{n}\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}(n^2) \\ &= \sum_{n=1}^{\infty} \frac{n}{(\sqrt{n^2+1}+n)} \\ &= 1/2\end{aligned}$$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n+1)-\ln(n)}{n}$ converges ($b_n = 1/n^2$, use L'Hospital to verify)

$$\begin{aligned}\lim_{n\to\infty} \frac{\ln(n+1)-\ln(n)}{n}(n^2) &= \lim_{n\to\infty} n\ln(1+1/n) \\ &= 1\end{aligned}$$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}-1}{n}$ converges ($b_n = 1/n^2$, use L'Hospital)

$$\begin{aligned}\lim_{n\to\infty} \frac{e^{1/n}-1}{n}(n^2) &= \lim_{n\to\infty} \frac{e^{1/n}-1}{1/n} \\ &= \lim_{n\to\infty} \frac{-1/n^2 e^{1/n}}{-1/n^2} \\ &= \lim_{n\to\infty} e^{1/n} \\ &= 1\end{aligned}$$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+\pi}$ converges ($b_n = 1/n^2$)

$$\begin{aligned}\lim_{n\to\infty} (\frac{1}{n} - \frac{1}{n+\pi})(n^2) &= \lim_{n\to\infty} \frac{n^2+n\pi}{n+\pi} - \frac{n^2}{n+\pi} \\ &= \lim_{n\to\infty} \frac{n\pi}{n+\pi} \\ &= \pi\end{aligned}$$

Bonus

1. Show that $1 + 1/1! + 1/2! + 1/3! + \ldots$ converges. (Hint: $3 \cdot 2 > 2 \cdot 2$, $4 \cdot 3 \cdot 2 > 2 \cdot 2 \cdot 2 \ldots$)
   Limit comparison with $b_n = 1/2^n$: $\lim_{n\to\infty} 2^n/n! = 0$, so $\sum_{n=0}^{\infty} 1/n!$ converges because $\sum_{n=0}^{\infty} 1/2^n$ converges.

2. For what values of x does the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?
   All values of x. (Limit comparison with $b_n = 1/2^n$ will again work)

3. If we integrate both sides of $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$, we get $-\ln(1-x) = x + x^2/2 + x^3/3 + \ldots$ (this actually works!) For what values of x does the series representing $-\ln(1-x)$ converge?
   The series converges for $-1 \le x < 1$.