11.3-11.4: Review

Monday, March 2

Recap

Determine whether each of the following series converge:

1. $\sum_{n=1}^{\infty} 1/n$
2. $\sum_{n=1}^{\infty} 1/n^2$
3. $\sum_{n=1}^{\infty} 1/\sqrt{n}$
4. $\sum_{n=1}^{\infty} (1/2)^n$
5. $\sum_{n=1}^{\infty} (-1/2)^n$
6. $\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n}$
7. $\sum_{n=1}^{\infty} (-\pi/3)^n$
8. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$
9. $\sum_{n=1}^{\infty} (-1)^n$

Limit Comparison Test

Determine whether each of the folowing series converge and find an appropriate function to use with the Limit Comparison Test:

1. $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$
2. $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+3}}{n^2}$
3. $\sum_{n=1}^{\infty} \frac{1+3^n}{2^n+\ln n}$
4. $\sum_{n=1}^{\infty} \frac{n^2+\sqrt{n^2+1}}{n^3+\sin n}$
5. $\sum_{n=1}^{\infty} \frac{n^3+\sqrt{n^3+1}}{\sqrt{n^7+\sin n}}$
6. $\sum_{n=1}^{\infty} \frac{n^2+2n+2^n}{3^n+n}$
7. $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{2n+1}$
8. $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{n}{n^2+6}$
9. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$
10. $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$
11. $\sum_{n=1}^{\infty} \frac{5^n}{n!}$
12. $\sum_{n=1}^{\infty} \frac{3}{3+\ln n}$

True or False?

If false, find a counterexample.

1. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, then $\sum_{n=1}^{\infty} a_n b_n$ converges.
3. If $\lim_{n\to\infty} a_n = 0$ but $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n b_n$ diverges.
4. If $\lim_{n\to\infty} a_n = 1$ and $b_n > 0, \sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n b_n$ converges.
5. If $a_n, b_n > 0$, $\lim_{n\to\infty} a_n/b_n = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
6. If $a_n > b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Tricks With Power Series

Convergence and limit problems can be simplified by replacing compicated terms with their power series (up to an appropriately high degree).

$f(x)$	Approximation when $x \approx 0$	Limits
$\sin(x)$	$x - x^3/3! + x^5/5! - \dots$	$\lim_{x\to 0} \frac{\sin x}{x} = 1$, $\lim_{x\to 0} \frac{\sin x - x}{x^3} = -1/6$
$\cos(x)$	$1 - x^2/2 + x^4/4! - \dots$	$\lim_{x\to 0} \frac{\cos(x) - 1}{x} = 0$, $\lim_{x\to 0} \frac{\cos(x) - 1}{x^2} = -1/2$
e^x	$1 + x + x^2/2 + x^3/3! + \dots$	$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$, $\lim_{x\to 0} \frac{e^x - x - 1}{x^2} = 1/2$
$\ln(1+x)$	$x - x^2/2 + x^3/3 - \dots$	$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$, $\lim_{x\to 0} \frac{\ln(1+x) - x}{x^2} = -1/2$
$\arctan(x)$	$x - x^3/3 + x^5/5 - \dots$	$\lim_{x\to 0} \frac{\arctan x}{x} = 1$
$\tan(x)$	$x + x^3/3 + 2x^5/15 + \dots$	$\lim_{x\to 0} \frac{\tan x}{x} = 1$

Example: Determine whether $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$ converges or diverges.

When n is large, $1/n \approx 0$ and so $\sin(1/n) \approx 1/n$. Therefore $\frac{\sin(1/n)}{\sqrt{n}} \approx \frac{1}{n^{3/2}}$. The series converges, and this can be shown by using the Limit Comparison Test with $\sum_{n=1}^{\infty} 1/n^{3/2}$.

Exercises:

1. Verify the limits in the table using L'Hospital's Rule.
2. What is $\lim_{n\to\infty} n^{1/n}$? (use $a^b = e^{b \ln a}$)
3. Show that if $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} \ln(1 + a_n)$ also converges. (use the Limit Comparison Test)

Decide whether each of the following series converge:

1. $\sum_{n=1}^{\infty} 1 - \cos(1/n)$
2. $\sum_{n=1}^{\infty} n^{1/n} - 1$
3. $\sum_{n=1}^{\infty} \sqrt{n} \sin \frac{\pi}{n^2}$
4. $\sum_{n=1}^{\infty} \frac{\ln(1 + 1/n)}{\sqrt{n}}$
5. $\sum_{n=1}^{\infty} \frac{e^{1/n} - 1}{n}$
6. $\sum_{n=1}^{\infty} \tan(1/n) \ln(1 + 1/n)$