# 11.3-11.4: Review

Monday, March 2

## Recap

Determine whether each of the following series converge:

1. $\sum_{n=1}^{\infty} 1/n$ diverges
2. $\sum_{n=1}^{\infty} 1/n^2$ converges
3. $\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges
4. $\sum_{n=1}^{\infty} (1/2)^n$ converges
5. $\sum_{n=1}^{\infty} (-1/2)^n$ converges
6. $\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n}$ converges (sub $u = \ln x$)
7. $\sum_{n=1}^{\infty} (-\pi/3)^n$ diverges
8. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges (sub $u = \ln x$)
9. $\sum_{n=1}^{\infty} (-1)^n$ diverges

## Limit Comparison Test

Determine whether each of the folowing series converge and find an appropriate function to use with the Limit Comparison Test:

1. $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ converges ($b_n = 1/n^3$)
2. $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+3}}{n^2}$ diverges ($b_n = 1/\sqrt{n}$)
3. $\sum_{n=1}^{\infty} \frac{1+3^n}{2^n + \ln n}$ diverges ($\lim_{n\to\infty} a_n \neq 0$)
4. $\sum_{n=1}^{\infty} \frac{n^2 + \sqrt{n^2+1}}{n^3 + \sin n}$ diverges ($b_n = 1/n$)
5. $\sum_{n=1}^{\infty} \frac{n^3 + \sqrt{n^3+1}}{\sqrt{n^7 + \sin n}}$ diverges ($b_n = 1/\sqrt{n}$)
6. $\sum_{n=1}^{\infty} \frac{n^2 + 2n + 2^n}{3^n + n}$ converges ($b_n = (2/3)^n$)
7. $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{2n+1}$ converges ($b_n = 1/n^{3/2}$)
8. $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{n}{n^2+6}$ converges ($b_n = 1/n^2$)
9. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges ($b_n = 1/n^{3/2}$)
10. $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$ converges ($b_n = (2/3)^n$)
11. $\sum_{n=1}^{\infty} \frac{5^n}{n!}$ converges ($b_n = (1/2)^n$)
12. $\sum_{n=1}^{\infty} \frac{3}{3 + \ln n}$ diverges ($b_n = 1/n$)

## True or False?

If false, find a counterexample.

1. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

   False: $a_n = 1/n$.

2. If $a_n, b_n > 0$ and $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

   True, since $\lim_{n\to\infty} a_n b_n / a_n = \lim_{n\to\infty} b_n = 0$, so the series converges by comparison with $\sum_{n=1}^{\infty} a_n$ (or $\sum_{n=1}^{\infty} b_n$).

3. If $\lim_{n\to\infty} a_n = 0$ but $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n b_n$ diverges.

   False: $a_n = 0$, or $a_n = 1/n^2, b_n = 1$.

4. If $\lim_{n\to\infty} a_n = 1$ and $b_n > 0, \sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

   True, by limit comparison with $\sum_{n=1}^{\infty} b_n$.

5. If $a_n, b_n > 0$, $\lim_{n\to\infty} a_n / b_n = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

   True by limit comparison.

6. If $a_n > b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

   False: $a_n = 1, b_n = 1/n^2$.

## Tricks With Power Series

Convergence and limit problems can be simplified by replacing compicated terms with their power series (up to an appropriately high degree).

| $f(x)$ | Approximation when $x \approx 0$ | Limits |
|---|---|---|
| $\sin(x)$ | $x - x^3/3! + x^5/5! - \dots$ | $\lim_{x\to 0} \frac{\sin x}{x} = 1, \lim_{x\to 0} \frac{\sin x - x}{x^3} = -1/6$ |
| $\cos(x)$ | $1 - x^2/2 + x^4/4! - \dots$ | $\lim_{x\to 0} \frac{\cos(x) - 1}{x} = 0, \lim_{x\to 0} \frac{\cos(x) - 1}{x^2} = -1/2$ |
| $e^x$ | $1 + x + x^2/2 + x^3/3! + \dots$ | $\lim_{x\to 0} \frac{e^x - 1}{x} = 1, \lim_{x\to 0} \frac{e^x - x - 1}{x^2} = 1/2$ |
| $\ln(1+x)$ | $x - x^2/2 + x^3/3 - \dots$ | $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1, \lim_{x\to 0} \frac{\ln(1+x) - x}{x^2} = -1/2$ |
| $\arctan(x)$ | $x - x^3/3 + x^5/5 - \dots$ | $\lim_{x\to 0} \frac{\arctan x}{x} = 1$ |
| $\tan(x)$ | $x + x^3/3 + 2x^5/15 + \dots$ | $\lim_{x\to 0} \frac{\tan x}{x} = 1$ |

**Example:** Determine whether $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$ converges or diverges.

When $n$ is large, $1/n \approx 0$ and so $\sin(1/n) \approx 1/n$. Therefore $\frac{\sin(1/n)}{\sqrt{n}} \approx \frac{1}{n^{3/2}}$. The series converges, and this can be shown by using the Limit Comparison Test with $\sum_{n=1}^{\infty} 1/n^{3/2}$.

**Exercises:**

1. Verify the limits in the table using L'Hospital's Rule.
2. What is $\lim_{n\to\infty} n^{1/n}$? (use $a^b = e^{b \ln a}$)
3. Show that if $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} \ln(1 + a_n)$ also converges. (use the Limit Comparison Test)

Decide whether each of the following series converge:

1. $\sum_{n=1}^{\infty} 1 - \cos(1/n)$ converges
2. $\sum_{n=1}^{\infty} n^{1/n} - 1$ diverges
3. $\sum_{n=1}^{\infty} \sqrt{n} \sin \frac{\pi}{n^2}$ converges
4. $\sum_{n=1}^{\infty} \frac{\ln(1 + 1/n)}{\sqrt{n}}$ converges
5. $\sum_{n=1}^{\infty} \frac{e^{1/n} - 1}{n}$ converges
6. $\sum_{n=1}^{\infty} \tan(1/n) \ln(1 + 1/n)$ converges