The Limit Comparison Test for Integrals

Say we want to prove that the integral $\displaystyle\int_1^\infty \frac{x^2}{3+x^3}\,dx$ diverges. To do this using the comparison test (and comparing to $1/x$), we would have to show that $\frac{x^2}{3+x^3}$ is eventually greater than C/x for some constant $C > 0$. There is no one right way to do this, but one possible answer is the following:
If $x \geq 1$ then $x^3 \geq 1$, so $3 \leq 3x^3$ and $3 + x^3 \leq 4x^3$. Therefore $\frac{1}{3+x^3} \geq \frac{1}{4x^3}$, and so $\frac{x^2}{3+x^3} \geq \frac{x^2}{4x^3} = \frac{1}{4}\left(\frac{1}{x}\right)$.
Other cases might require us to split the integral into a finite part and an infinite part. For example, to show that $\int_0^\infty e^{-x^2}\,dx$ converges we could write it as $\int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$. The first integral is finite since e^{-x^2} is a bounded function and the second is bounded above by $\int_1^\infty e^{-x}\,dx$ (Splitting the integral is necessary here since e^{-x^2} is not less than or equal to e^{-x} on [0,1], and proving that any particular constant works for $[0, \infty]$ is somewhat more difficult.)
Making strict comparisons such as these can be tedious. Fortunately, there is a simpler method that doesn't require us to guess at a constant C or decide how to split the integral:

Theorem 0.1 *Limit Comparison Test*
For two bounded functions $f(x), g(x)$, suppose that $\lim_{x\to\infty} f(x)/g(x) = C$ for some $0 < C < \infty$. Then the integrals $\int_0^\infty f(x)\,dx$ and $\int_0^\infty g(x)\,dx$ either both converge or both diverge.

Proof: By the delta-epsilon definition of $\lim_{x\to\infty} f(x)/g(x) = C$, for every $\epsilon > 0$ there exists N such that if $x > N$ then $|f(x)/g(x) - C| < L$. Picking $\epsilon = C/2$, there therefore exists a number N such that if $x > N$ then

$$-C/2 < f(x)/g(x) - C < C/2$$
$$C/2 < f(x)/g(x) < 3C/2$$
$$\frac{C}{2}g(x) < f(x) < \frac{3C}{2}g(x)$$

Then split the integral $\int_0^\infty f(x)\,dx$ into $\int_0^N f(x)\,dx + \int_N^\infty f(x)\,dx$. The first of the two integrals must converge since $f(x)$ is a bounded function (and similarly for $\int_0^N g(x)\,dx$). If $\int_N^\infty g(x)\,dx$ converges, then $\int_N^\infty f(x)\,dx < \frac{3C}{2}\int_N^\infty g(x)\,dx$ converges by the comparison test. If $\int_N^\infty g(x)\,dx$ diverges, then $\int_N^\infty f(x)\,dx > \frac{C}{2}\int_N^\infty g(x)\,dx$ diverges by the comparison test.
Therefore, $\int_0^\infty f(x)\,dx$ converges if $\int_0^\infty g(x)\,dx$ converges and diverges if $\int_0^\infty g(x)\,dx$ diverges.

Corollary 0.1.1 *Limit Comparison Test with* $1/x^p$
If $f(x)$ is bounded and $\lim_{x\to\infty} f(x)x^p = C$ for some constant $0 < C < \infty$, then $\int_0^\infty f(x)\,dx$ converges if $\int_0^\infty 1/x^p\,dx$ converges and diverges if $\int_0^\infty 1/x^p\,dx$ diverges.

Examples

1. Since $\displaystyle\lim_{x\to\infty}\left(\frac{x^2}{3+x^3}\right)\cdot x = \lim_{x\to\infty}\frac{x^3}{3+x^3} = \lim_{x\to\infty}\frac{1}{1+3/x^3} = 1$, and $\displaystyle\int_0^\infty \frac{1}{x}\,dx$ diverges, $\displaystyle\int_0^\infty \frac{x^2}{3+x^3}$ also diverges.

2. Since $\lim_{x\to\infty} e^{-x^2}\cdot x^2 = \lim_{x\to\infty} e^x/e^{x^2} = \lim_{x\to\infty} e^{-x^2+x} = 0$ and $\int_0^\infty e^{-x}\,dx$ converges, $\int_0^\infty e^{-x^2}\,dx$ also converges.

This test will also work for integrating functions that tend to infinity at a specific point.

Theorem 0.2 *Limit Comparison Test II*
For two functions $f(x)$ and $g(x)$ that are bounded except at 0, if $\lim_{x\to 0} f(x)/g(x) = C$ for some constant $0 < C < \infty$, then the integrals $\int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ will either both converge or both diverge.

Corollary 0.2.1 *Test II with* $1/x^p$
If $f(x)$ is bounded except possibly at 0 and $\lim_{x\to 0} f(x)x^p = C$ for some constant $0 < C < \infty$, then $\int_0^1 f(x)\,dx$ converges if $\int_0^1 1/x^p\,dx$ converges and diverges if $\int_0^1 1/x^p\,dx$ diverges.

Example: The integral $\displaystyle\int_0^1 \frac{x+3}{x(x-2)}\,dx$ diverges since $\displaystyle\lim_{x\to 0}\left(\frac{x+3}{x(x-2)}\right)x = \lim_{x\to 0}\frac{x+3}{x-2}\cdot\frac{x}{x} = -3/2$ and $\displaystyle\int_0^1 \frac{1}{x}\,dx$ diverges.