Review: Chapters 9 & 17

Monday, May 4

9.1-9.2: Differential Equations and Direction Fields

1. Show that the function $y = x \sin x$ satisfies the differential equation $y'' + y = 2\cos x$.

$$\begin{aligned} y'' + y &= (x \sin x)'' - x \sin x \\ &= (\sin x + x \cos x)' + x \sin x \\ &= \cos x + \cos x - x \sin x + x \sin x \\ &= 2 \cos x \end{aligned}$$

2. Show that the function $y = xe^{-x} + 2$ satisfies the differential equation $y - xy' = x^2e^{-x} + 2$.

$$\begin{aligned} y - xy' &= xe^{-x} + 2 - x(xe^{-x} + 2)' \\ &= xe^{-x} + 2 - x(e^{-x} - xe^{-x}) \\ &= xe^{-x} + 2 - xe^{-x} + x^2e^{-x} \\ &= x^2e^{-x} + 2 \end{aligned}$$

3. If $y' = y$ and $y(0) = 1$, estimate $y(1)$ using Euler's method with a step size of $\Delta x = 0.5$.

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 1 \\ y(0.5) &\approx y(0) + \Delta x \cdot y'(0) \\ y(0.5) &\approx 1.5 \\ y'(0.5) &= 1.5 \\ y(1) &\approx y(0.5) + \Delta x \cdot y'(0.5) \\ y(1) &\approx 1.5 + 0.5 \cdot 1.5 \\ y(1) &\approx 2.25 \end{aligned}$$

The solution to the given differential equation is $y = e^x$, and so $y(1) = e \approx 2.718$. To get a better approximation from Euler's method we would have to use a much smaller step size.

4. Match the following differential equations with the appropriate direction fields:

(a) $y' = x + y - 1$

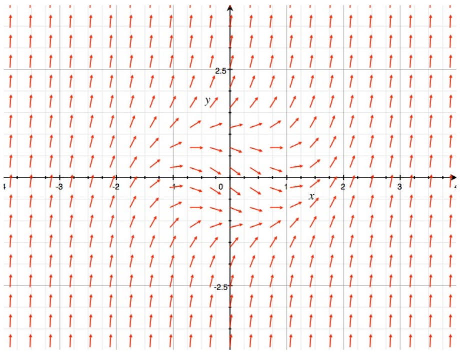
(b) $y' = xy - 1$

(c) $y' = x^2 + y^2 - 1$

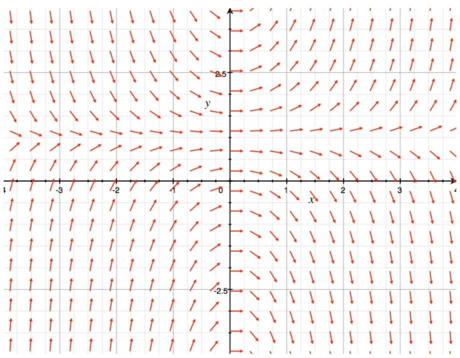
(d) $y' = x - y + 1$

(e) $y' = x(y - 1)$

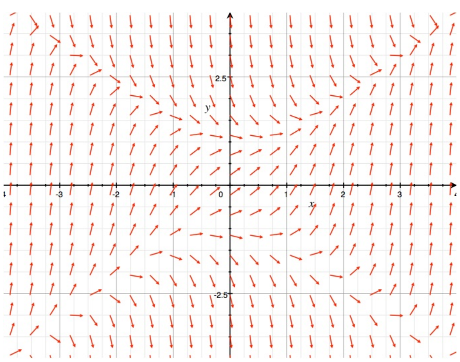
(f) $y' = x^2 - y^2 + 1$



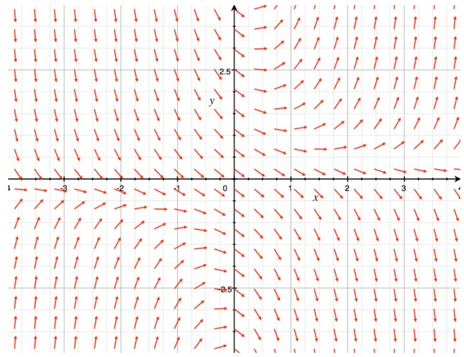
1. C



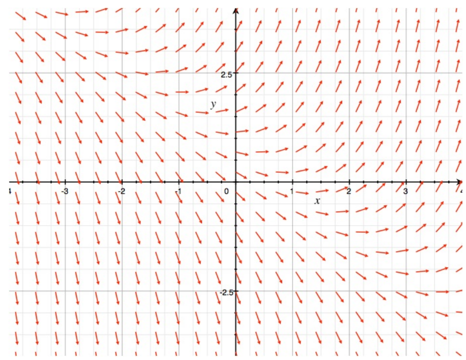
2. E



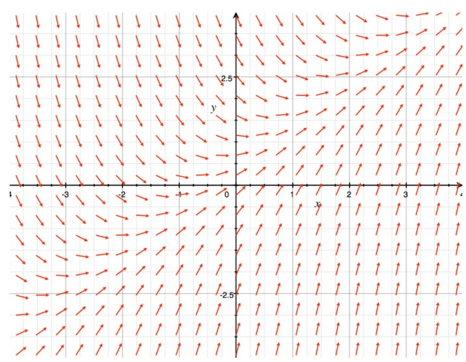
3. F



4. B



5. A



6. D

9.3: Separable Equations

1. $y' = x/y, y(0) = -3$

$$\begin{aligned}
\frac{dy}{dx} &= x/y \\
y\,dy &= x\,dx \\
\int y\,dy &= \int x\,dx \\
\frac{1}{2}y^2 &= \frac{1}{2}x^2 + C \\
y &= \pm\sqrt{x^2 + C} \\
y &= -\sqrt{x^2 + 9}
\end{aligned}$$

2. $y' = xy\sin x, y(0) = 1$

$$\begin{aligned}
\frac{dy}{dx} &= xy\sin x \\
\frac{1}{y}\,dy &= x\sin x\,dx \\
\int \frac{1}{y}\,dy &= \int x\sin x\,dx \\
\ln y &= -x\cos x + \sin x + C \\
y &= e^{-x\cos x + \sin x + C} \\
y &= e^{-x\cos x + \sin x}
\end{aligned}$$

3. $xy' - y = 1, y(2) = 3$

$$\begin{aligned}
x\frac{dy}{dx} &= 1 + y \\
\frac{dy}{1+y} &= \frac{dx}{x} \\
\ln(1+y) &= \ln x + C \\
1 + y &= e^{\ln x + C} \\
y &= Kx - 1 \\
y &= 2x - 1
\end{aligned}$$

9.5: Linear Equations

1. $xy' - y = 1, y(2) = 3$

$$
\begin{aligned}
y' - \frac{1}{x}y &= \frac{1}{x} \\
I &= e^{\int -1/x} \\
&= e^{-\ln x} \\
&= 1/x \\
y &= (\int IQ)/I \\
&= x(\int 1/x^2) \\
&= x(-1/x + C) \\
&= Cx - 1 \\
y &= 2x - 1
\end{aligned}
$$

Unsurprisingly, this is the same answer as for the previous problem.

2. $2xy' + y = 6x, y(4) = 20$

$$
\begin{aligned}
y' + \frac{1}{2x}y &= 3 \\
I &= e^{\int 1/2x} \\
&= e^{\frac{1}{2}\ln x} \\
&= \sqrt{x} \\
y &= (\int IQ)/I \\
&= \frac{1}{\sqrt{x}}(\int 3\sqrt{x}) \\
&= \frac{1}{\sqrt{x}}(C + 2x^{3/2}) \\
&= 2x + C/\sqrt{x} \\
y &= 2x + 24/\sqrt{x}
\end{aligned}
$$

3. $y' + xy = x, y(1) = 1.$

$$
\begin{aligned}
I &= e^{\int x} \\
&= e^{x^2/2} \\
y &= (\int IQ)/I \\
&= e^{-x^2/2}(\int xe^{x^2/2}) \\
&= e^{-x^2/2}(e^{x^2/2} + C) \\
&= 1 + Ce^{-x^2/2} \\
y &= 1
\end{aligned}
$$

17.1-2: Second-order Linear Equations

Solve each non-homogeneous equation using either variation of parameters or the method of undetermined coefficients, whichever is more appropriate.

1. $y'' + 3y' + 2y = \sin x + 2\cos x$

   Undetermined coefficients:

$$\begin{aligned} y &= A\sin x + B\cos x \\ y' &= A\cos x - B\sin x \\ y'' &= -A\sin x - B\cos x \\ y'' + 3y' + 2y &= (A - 3B)\sin x + (3A + B)\cos x \\ (A - 3B) &= 1 \\ 3A + B &= 2 \\ A &= 7/10 \\ B &= -1/10 \\ y_p &= \frac{7}{10}\sin x - \frac{1}{10}\cos x \\ y_c &= C_1 e^{-x} + C_2 e^{-2x} \\ y &= \frac{7}{10}\sin x - \frac{1}{10}\cos x + C_1 e^{-x} + C_2 e^{-2x} \end{aligned}$$

2. $y'' - 2y' + y = 2xe^x - e^x$

   The general solution to $y'' - 2y' + y = 0$ is $y_c = C_1 e^x + C_2 x e^x$, so we'll have to try multiplying by x or x^2:

$$\begin{aligned} y &= Ax^3 e^x + Bx^2 e^x \\ y' &= Ax^3 e^x + (3A + B)x^2 e^x + 2Bxe^x \\ y'' &= Ax^3 e^x + (6A + B)x^2 e^x + (6A + 4B)xe^x + 2Be^x \\ y'' - 2y' + y &= 6Axe^x + 2Be^x \\ A &= 1/3 \\ B &= -1/2 \\ y &= \frac{1}{3}x^3 e^x - \frac{1}{2}x^2 e^x + C_1 e^x + C_2 x e^x \end{aligned}$$

3. $y'' + y = 2\sin x + 3$

   The general solution to $y'' + y = 0$ is $y = A\sin x + B\cos x$, so try multiplying by x to solve for $y'' + y = 2\sin x$. Separately, try $y = Cx^2 + Dx + E$ to solve $y'' + y = 3$.

$$
\begin{aligned}
y &= Ax\sin x + Bx\cos x + Cx^2 + Dx + E \\
y' &= A\sin x + Ax\cos x + B\cos x - Bx\sin x + 2Cx + D \\
y'' &= 2A\cos x - Ax\sin x - 2B\sin x - Bx\cos x + 2C \\
y'' + y &= -2B\sin x + 2A\cos x + Cx^2 + Dx + (2C + E) \\
B &= -1 \\
A &= 0 \\
C &= 0 \\
D &= 0 \\
E &= 3 \\
y_p &= -x\cos x + 3 \\
y &= -x\cos x + 3 + A\sin x + B\cos x
\end{aligned}
$$

4. $y'' + y = \frac{1}{\cos x}$

$$
\begin{aligned}
y_1, y_2, &= \cos x, \sin x \\
W &= 1 \\
y &= -\cos x \int \frac{1}{\cos x}\sin x + \sin x \int 1 \\
&= -\cos x(-\ln\cos x + C_1) + \sin x(x + C_2) \\
&= x\sin x + \cos x \ln\cos x + C_1\cos x + C_2\sin x
\end{aligned}
$$

5. $y'' + 3y' + 2y = \sin(e^x)$

$$
\begin{aligned}
y_1, y_2, &= e^{-x}, e^{-2x} \\
W &= -e^{-3x} \\
y &= -e^{-x} \int -e^x \sin e^x + e^{-2x} \int -e^{2x} \sin e^x \\
&= e^{-x}(-\cos e^x + C_1) + e^{-2x}(e^x \cos e^x - \sin e^x + C_2) \\
&= -e^{-2x}\sin e^x + C_1 e^{-x} + C_2 e^{-2x}
\end{aligned}
$$