Review: Chapter 7

Wednesday, April 29

Chapter 7: Integrals

7.1: Integration by Parts

1. $\int x \sin x \, dx = -x \cos x + \sin x$
2. $\int x e^x \, dx = xe^x - e^x$
3. $\int \ln x \, dx = x \ln x - x$
4. $\int \arctan x \, dx = x \arctan x - \frac{1}{2} \ln(1 + x^2)$
5. $\int x e^{x^2} \, dx = \frac{1}{2} e^{x^2}$
6. $\int x^3 \sin(x^2) \, dx = -\frac{1}{2} x^2 \cos(x^2) + \frac{1}{2} \sin(x^2)$

7.2: Trig Integrals

1. $\int \sin^2 x \, dx = \int \frac{1 - \cos(2x)}{2} = \frac{x}{2} - \frac{1}{2} \sin 2x$
2. $\int \sin x \cos x \, dx = \int \frac{1}{2} \sin 2x = \frac{-1}{4} \cos 2x (+C)$
3. OR: $\int \sin x \cos x \, dx = -\frac{1}{2} \cos^2 x (+C)$
4. $\int \sin 2x \sin 4x \, dx = \int 2 \sin^2(2x) \cos 2x \, dx = \int \frac{1}{3} \sin^3(2x)$
5. $\int \tan x \sec^3 x \, dx = \int \sec^2 x (\sec x \tan x \, dx) = \frac{1}{3} \sec^3 x$
6. $\int \cos^2 x \tan^3 x \, dx = \int \frac{\sin^3 x}{\cos x} = \int \frac{\cos^2 x - 1}{\cos x} (-\sin x dx) = \frac{1}{2} \cos^2 x - \ln \cos x$
7. $\int \tan^3 x \, dx = \int (\sec^2 x - 1) \tan x \, dx = \int \sec x (\sec x \tan x dx) - \int \tan x \, dx = \frac{1}{2} \sec^2(x) + \ln \cos x$

7.3: Trig Substitutions

1. $\int x^3 \sqrt{1 - x^2} \, dx = \int \sin^3 \theta \cos^2 \theta \, d\theta = \int (\cos^2 \theta - 1) \cos \theta (-\sin \theta \, d\theta) = \int y^4 - y^2 \, dy = \frac{1}{5} y^5 - \frac{1}{3} y^3 = \frac{1}{5} (1 - x^2)^{5/2} - \frac{1}{3} (1 - x^2)^{3/2}$

2. $\int \frac{\sqrt{1+x^2}}{x}\,dx = \int \frac{\sec^3\theta}{\tan\theta}\,d\theta = \int \frac{(\tan^2\theta+1)\sec\theta}{\tan\theta}\,d\theta = \int \sec\theta\tan\theta\,d\theta + \int \csc\theta = \ldots = \sqrt{x^2+1} - \ln\frac{1+\sqrt{1+x^2}}{x}$

3. $\int \frac{1}{x^2\sqrt{x^2-1}}\,dx = \int \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\tan\theta} = \int \cos\theta\,d\theta = \sin\theta = \frac{\sqrt{x^2-1}}{x}$

4. $\int \frac{x}{\sqrt{x^2+x+1}}\,dx$ gets ugly. The first thing to do would be to say $x^2+x+1 = (x+1/2)^2+3/4$, then sub $x+1/2 = \tan\theta$.

5. $\int \sqrt{x^2+2x}\,dx$ also gets ugly. But the first thing to try would be $x^2+2x = (x+1)^2-1$, then sub $x+1 = \sec\theta$.

6. $\int \frac{1}{\sqrt{x^2-6x+13}}\,dx$ also ugly (it's $\sinh^{-1}(\frac{x-3}{2})$), but completing the square would give $x^2-6x+13 = (x-3)^2+4$, then try the substitution $x-3 = 2\tan\theta$.

7.4: Partial Fractions

1. $\int \frac{x}{1+x^2} = \frac{1}{2}\ln(1+x^2)$

2. $\int \frac{x+1}{x^2-1} = \int \frac{1}{x-1} = \ln(x-1)$

3. $\int \frac{x^2+2x+1}{(x+3)^2} = x - \frac{4}{x+3} - 4\ln(x+3)$

4. $\int \frac{x^2+3}{(x+1)(x+2)} = x + 4\ln(x+1) - 7\ln(x+2)$

5. $\int \frac{x+5}{(x+1)(x-2)^2} = -\frac{7}{x-2} - 6\ln(x-2) + 6\ln(x-1)$

6. $\int \frac{x+1}{x^2+2x+2} = \frac{1}{2}\ln(x^2+2x+2)$

7.7: Approximate Integration

1. Estimate the value of π by approximating $\int_0^1 \frac{1}{1+x^2}\,dx$ with the Midpoint, Trapezoidal, and Simpson's Rule with $n=4$.

2. Put error bounds on each estimate.

7.8: Improper Integrals

Decide whether the following integrals converge or diverge. If the integral converges, evaluate it.

1. $\int_0^1 1/x^2\,dx$: divergent

2. $\int_0^\infty e^{-x}\,dx = 1$

3. $\int_0^1 1/x\, dx$: divergent

4. $\int_0^2 1/\sqrt{x}\, dx = 2\sqrt{2}$

5. $\int_1^\infty 1/\sqrt{x}\, dx$: divergent

6. $\int_1^\infty 1/x^2\, dx = 1$

7. $\int_1^\infty \frac{1}{(x-1)x^2}\, dx$: divergent due to the singularity at $x = 1$.

8. $\int_0^1 \ln x\, dx = \lim_{\epsilon \to 0} x \ln x - x|_\epsilon^1 = -1 - \lim_{\epsilon \to 0} \epsilon \ln \epsilon = -1$

9. $\int_2^4 1/\sqrt{|x-3|} = \int_2^3 1/\sqrt{|x-3|} + \int_3^4 1/\sqrt{|x-3|} = 2 + 2 = 4$