Review: Chapter 11

Friday, May 1

11.1: Sequences

True or False! As always, give a counterexample to the false statements.

1. If $\{a_n\}$and $\{b_n\}$are convergent then $\{a_n + b_n\}$is convergent.

   True.

2. If $\{a_n\}$and $\{b_n\}$are convergent then $\{a_n b_n\}$is convergent.

   True.

3. If $\{a_n\}$and $\{b_n\}$are divergent then $\{a_n + b_n\}$is divergent.

   False: $a_n = n, b_n = -n, a_n + b_n = 0$.

4. If $\{a_n\}$and $\{b_n\}$are divergent then $\{a_n b_n\}$is divergent.

   False: $a_n = b_n = (-1)^n, a_n b_n = 1$.

5. If f is continuous and $\{a_n\}$converges then $\lim_{n\to\infty} f(a_n)$ exists.

   True, and furthermore $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$.

6. If f is continuous and $\{a_n\}$diverges then $\lim_{n\to\infty} f(a_n)$ does not exist.

   False: If $f(x) = 0$ for all x then $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} 0 = 0$ regardless of the sequence $\{a_n\}$.

Not True or False!

1. Define what it means for a sequence to be bounded.

   There exists M such that $|a_n| \leq M$ for all n.

2. What are the conditions for the Monotone Convergence Theorem?

   MTC: if a_n is monotonic (either increasing or decreasing) and bounded then $\{a_n\}$ converges.

   The rationale is that a sequence converges either if it is increasing and bounded above or if it is decreasing and bounded below.

3. Give an example of a monotonic sequence that does not converge.

   $a_n = n$

4. Give an example of a bounded sequence that does not converge.

   $a_n = (-1)^n$

11.2: Series

1. What is the harmonic series? Does it converge or diverge?

   The partial sums of the harmonic series $h_k = \sum_{n=1}^{k} \frac{1}{n}$. The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges due to the Integral Test (comparison with $\int_0^\infty \frac{1}{x}\,dx$).

2. Decide whether $\displaystyle\sum_{n=1}^{\infty} 3 \cdot 2^n$ and $\displaystyle\sum_{n=1}^{\infty} 3/2^n$ converge or diverge. Find the limits if they converge.

   The first series diverges since $|2| > 1$. For the second $\sum_{n=1}^{\infty} 3/2^n = \frac{3}{2}\sum_{n=0}^{\infty}(1/2)^n = \frac{3}{2}\frac{1}{1-1/2} = 3$.

11.3-11.7: Lots of Convergence Tests

Converge or diverge?

1. $\sum_{n=1}^{\infty} \frac{3^n}{n!}$: converge
2. $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$: diverge
3. $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{n^2}$: diverge
4. $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$: converge
5. $\sum_{n=1}^{\infty} \frac{\ln n}{n^{1.1}}$: converge
6. $\sum_{n=1}^{\infty} \frac{n^{30}}{1.01^n}$: converge

For each of the following tests, do the following:

1. State what the test is.
2. Give an example of a series where the test proves convergence, if applicable.
3. Give an example of a series where the test proves divergence, if applicable.
4. Give an example of a series where the test is inconclusive or does not apply.

- Test For Divergence
  1. If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.
  2. Not applicable: this test can only show that a series diverges.
  3. $a_n = 1$, $a_n = 1 - 1/n$, anything where $\lim_{n\to\infty} a_n \neq 0$.
  4. If $\lim_{n\to\infty} a_n = 0$ the test is inconclusive: the sequences $1/n$ and $1/n^2$ both converge to zero but the first series diverges and the second converges.
- P-series Test
  1. $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$ and diverges otherwise.
  2. $\sum_{n=1}^{\infty} 1/n^2$ converges.
  3. $\sum_{n=1}^{\infty} 1/n$ and $\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverge.
  4. Does not apply directly to $1/\ln n$ or $1/(n^2+n+1)$, for example. You have to apply a comparison test first.
- Comparison Test
  1. If $a_n \geq b_n \geq 0$ and $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges. If $b_n \geq a_n \geq 0$ and $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.
  2. $\sum_{n=1}^{\infty}(2+\cos n)/n^2$ converges by comparison with $3/n^2$.
  3. $\sum_{n=1}^{\infty}(2+\cos n)/n$ diverges by comparison with $1/n$.
  4. Does not apply to functions such as $\sin n/n$ that have both positive and negative terms.
- Limit Comparison Test
  1. Also requires $a_n, b_n \geq 0$ (or $a_n, b_n \leq 0$ for all n). If $\lim_{n\to\infty} a_n/b_n = C < \infty$ and $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges too. If $\sum_{n=1}^{\infty} a_n/b_n = C > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges too.
  2. $\sum_{n=1}^{\infty}(n+\sin n)/(n^3)$ converges by limit comparison with $1/n^2$
  3. $\sum_{n=1}^{\infty}(n-\ln n)/n^2$ diverges by limit comparison with $1/n$.
  4. Like the comparison test, LCT does not apply to functions that have both positive and negative terms.
- Alternating Series Test

1. If a_n is alternating, if $|a_{n+1}| \leq |a_n|$ for all large enough n, and $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.
2. $(-1)^n/n$ converges.
3. Cannot be used to prove divergence.
4. Does not apply to $\sin(n)/n$ since it is not strictly alternating. Does not apply to $(1+2\cdot(-1)^n)/n$ since the terms are not decreasing in magnitude.

- Ratio Test

  1. Let $R = \lim_{n\to\infty} |a_{n+1}|/|a_n|$. If $R < 1$ then the series converges. If $R > 1$ the series diverges. If $R = 1$ the test is inconclusive.
  2. $\sum_{n=1}^{\infty} n^2/2^n$ converges.
  3. $\sum_{n=1}^{\infty} n!/5^n$ diverges (not that if $R > 1$ then the terms in the series do not even converge to zero).
  4. Inconclusive for $1/n$, n^2, $\ln(n)/n^5$, and in general all combinations of polynomial and logarithmic functions. Use only when exponentials and factorials appear.

- Root Test

  1. Let $R = \lim_{n\to\infty} \sqrt[n]{|a_n|}$. If $R < 1$ then the series converges. If $R > 1$ the series diverges. If $R = 1$ the test is inconclusive.
  2. The Root test gives the exact same results as the Ratio test. It is in general less useful; only use it for series with the form $\sum_{n=1}^{\infty}(G(n))^n$ for complicated functions G.

True/False!

1. If $\sum_{n=1}^{\infty} a_n$ is convergent then it is absolutely convergent.

   False: $a_n = 1/n$.

2. If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then it is convergent.

   True.

3. If the Ratio Test says that $\sum_{n=1}^{\infty} a_n$ converges then it converges absolutely.

   True, since the results of the Ratio test only depend on the absolute values $|a_n|$. This means that for a power series conditional convergence can only happen at the endpoints of the interval.

4. If $\sum_{n=1}^{\infty} a_n$ converges but the Ratio Test is inconclusive then $\sum_{n=1}^{\infty} a_n$ converges conditionally.

   False: $1/n^2$ converges absolutely both at -1 and 1.

5. If $\sum_{n=1}^{\infty} a_n$ is an alternating series then it converges.

   False: $a_n = (-1)^n$.

11.8-10: Taylor Series

Find the Taylor series for the following functions up to the x^5 term:

1. $\sin x = x - x^3/3! + x^5/5! - \ldots$
2. $\cos x = 1 - x^2/2! + x^4/4! - \ldots$
3. $e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + x^5/5! + \ldots$
4. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots$

5. $e^x \cos x = 1 + x - x^3/3 - x^4/6 - x^5/30 + \dots$

6. $\frac{x^2}{1+2x} = x^2 - 2x^3 + 4x^4 - 8x^5 + \dots$

Find power series that have the following radii of convergence:

1. $[-1, 1] : \sum_{n=1}^{\infty} x^n/n^2$

2. $(3, 5) : \sum_{n=1}^{\infty} (x-4)^n$

3. $[2, 3) : \sum_{n=1}^{\infty} 2^n (x - 5/2)^n = \sum_{n=1}^{\infty} (2x-5)^n$

4. $(0, 7] : \sum_{n=1}^{\infty} (-2/7)^n (x - 7/2)^n$

5. $(-\infty, \infty) : \sum_{n=1}^{\infty} x^n/n!$

6. $\{4\} : \sum_{n=1}^{\infty} n! x^n$