11.5: Alternating Series

Wednesday, March 4

True or False?

For all of these problems assume that $a_n, b_n > 0$ for all natural numbers n. If the statement is false, find a pair of functions (a_n, b_n) that serve as a counterexample.

1. If $a_n < b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $a_n < b_n$ for all $n > 1000$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
3. If $a_n > b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.
4. If $\lim_{n\to\infty} a_n/b_n = 5$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
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6. If $\lim_{n\to\infty} a_n/b_n = 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ converges.
7. If $\lim_{n\to\infty} a_n/b_n = \infty$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
8. $\ln(n) < \sqrt[3]{n}$ for all natural numbers n.

Suppose I claim that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge then $\sum_{n=1}^{\infty} a_n/b_n$ converges. Decide whether each of the following pairs of sequences is an example of the claim, a counterexample, or irrelevant:

1. $a_n = 1/n, b_n = 1/n^3$
2. $a_n = 1/n^4, b_n = 1/n^2$
3. $a_n = 1/n^3, b_n = 1/2^n$.
4. $a_n = 1/n^2, b_n = 1/n^2$
5. $a_n = 1/2^n, b_n = 1/n^2$
6. $a_n = 1/n^2, b_n = 1/n$

Terms Sometimes Cancel

Decide whether the following series converge or diverge:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} - \frac{n}{n^2+n+3}$
2. $\displaystyle\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} - \frac{\pi}{n+\pi}$
4. $\displaystyle\sum_{n=1}^{\infty} \sqrt{n^3+1} - \sqrt{n^3}$

Limit Comparison Test

Decide whether each of the following series converge or diverge, and find an appropriate sequence b_n to use with the Limit Comparison Test.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 \ln n}{3^n}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{1.002^n}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln^5(n)}{n^{1.002}}$

Alternating Series and Absolute Convergence

1. If $s = \sum_{n=1}^{\infty}(-1)^n b_n$ is a series with

   (a) $b_{n+1} \leq b_n$ for all n (or $n > N$, for some N)

   (b) $\lim_{n\to\infty} b_n = 0$

   then the series is convergent.

2. $|R_n| = |s - s_n| \leq b_{n+1}$.

3. If $s_n \geq s_{n+1}$, then $s_{n+1} \leq s \leq s_n$.

4. (Absolute Convergence) If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Some Series of Note

1. $\sin(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots$. Show that this power series converges for all x.

2. $\cos(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots$. Show that this power series converges for all x.

3. $\ln(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + \ldots$ Show that $1 - 1/2 + 1/3 - 1/4 + \ldots$ converges and is equal to $\ln(2)$.

4. Can you use the same power series to find $\ln(3)$? How about $\ln(0)$ or $\ln(2.001)$?

5. $\arctan(x) = x - x^3/3 + x^5/5 - x^7/7 + \ldots$ Show that $1 - 1/3 + 1/5 - 1/7 + \ldots$ converges and conclude that it is equal to $\pi/4$.

6. Prove that $\sum_{n=1}^{\infty} \cos(n)/n^2$ converges.

7. Can you tell whether $\sum_{n=1}^{\infty} \cos(n)/n$ converges?