11.5: Alternating Series

Wednesday, March 4

True or False?

For all of these problems assume that $a_n, b_n > 0$ for all natural numbers n. If the statement is false, find a pair of functions (a_n, b_n) that serve as a counterexample.

1. If $a_n < b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

   Answer: True.

2. If $a_n < b_n$ for all $n > 1000$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

   Answer: True.

3. If $a_n > b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

   Answer: True.

4. If $\lim_{n\to\infty} a_n/b_n = 5$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

   Answer: True.

5. If $\lim_{n\to\infty} a_n/b_n = 5$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

   Answer: True.

6. If $\lim_{n\to\infty} a_n/b_n = 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ converges.

   Answer: False. $a_n = 1/n, b_n = 1$.

7. If $\lim_{n\to\infty} a_n/b_n = \infty$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

   Answer: False. $a_n = n, b_n = 1/n^2$.

8. $\ln(n) < \sqrt[3]{n}$ for all natural numbers n.

   Answer: False. It holds true for $n \geq 94$ and $n \leq 6$, but any value of n in between will serve as a counterexample.

Suppose I claim that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge then $\sum_{n=1}^{\infty} a_n/b_n$ converges. Decide whether each of the following pairs of sequences is an example of the claim, a counterexample, or irrelevant:

1. $a_n = 1/n, b_n = 1/n^3$: irrelevant, since a_n does not converge.
2. $a_n = 1/n^4, b_n = 1/n^2$: example
3. $a_n = 1/n^3, b_n = 1/2^n$: counterexample
4. $a_n = 1/n^2, b_n = 1/n^2$: counterexample
5. $a_n = 1/2^n, b_n = 1/n^2$: example
6. $a_n = 1/n^2, b_n = 1/n$: irrelevant, since b_n does not converge.

Terms Sometimes Cancel

Decide whether the following series converge or diverge:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} - \frac{n}{n^2+n+3}$ converges
2. $\displaystyle\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n}$ diverges
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} - \frac{\pi}{n+\pi}$ diverges (the highest order terms do not cancel out fully)
4. $\displaystyle\sum_{n=1}^{\infty} \sqrt{n^3+1} - \sqrt{n^3}$ converges

Limit Comparison Test

Decide whether each of the following series converge or diverge, and find an appropriate sequence b_n to use with the Limit Comparison Test.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$: diverges ($b_n = 1/n$)
2. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$: converges ($b_n = \sqrt{n}/n^2 = 1/n^{3/2}$)
3. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$: converges ($b_n = (1.5/2)^n = (3/4)^n$)
4. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 \ln n}{3^n}$: converges ($b_n = (2/3)^n$)
5. $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{1.002^n}$: converges ($b_n = (1.001/1.002)^n$)
6. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln^5(n)}{n^{1.002}}$ converges ($b_n = n^{0.001}/n^{1.002} = \frac{1}{n^{1.001}}$)

Alternating Series and Absolute Convergence

1. If $s = \sum_{n=1}^{\infty} (-1)^n b_n$ is a series with
   (a) $b_{n+1} \leq b_n$ for all n (or $n > N$, for some N)
   (b) $\lim_{n\to\infty} b_n = 0$

   then the series is convergent.
2. $|R_n| = |s - s_n| \leq b_{n+1}$.
3. If $s_n \geq s_{n+1}$, then $s_{n+1} \leq s \leq s_n$.
4. (Absolute Convergence) If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Some Series of Note

1. $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$. Show that this power series converges for all x.

   The given series converges because 1) it is alternating (both for $x > 0$ and $x < 0\ldots$check this!), 2) its terms are EVENTUALLY decreasing in magnitude for any fixed x (since factorial growth outweighs exponential growth), and 3) its terms have a limit of zero as $n \to \infty$, so the Alternating Series test applies.

2. $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$. Show that this power series converges for all x.

   The given series converges because 1) it is alternating, 2) its terms are EVENTUALLY decreasing in magnitude for any fixed x (since factorial growth outweighs exponential growth), and 3) its terms have a limit of zero as $n \to \infty$, so the Alternating Series test applies.

3. $\ln(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + \ldots$ Show that $1 - 1/2 + 1/3 - 1/4 + \ldots$ converges and is equal to $\ln(2)$.

   The given series converges because 1) it is alternating, 2) its terms are strictly decreasing in magnitude, and 3) its terms have a limit of zero as $n \to \infty$, so the Alternating Series test applies.

   The limit of the series is $\ln(2)$ since the series comes from plugging $x = 1$ into the power series formula for $\ln(1 + x)$.

4. Can you use the same power series to find $\ln(3)$? How about $\ln(0)$ or $\ln(2.001)$?

   Nope, nope, and nope. These would correspond to $\ln(1 + 2), \ln(1 + (-1))$, and $\ln(1 + 1.001)$. The first and third have exponential growth in the numerator, so the limit of the terms in the series is not zero. The second series yields the Harmonic series $1 + 1/2 + 1/3 + 1/4 + \ldots$, which does not converge.

5. $\arctan(x) = x - x^3/3 + x^5/5 - x^7/7 + \ldots$ Show that $1 - 1/3 + 1/5 - 1/7 + \ldots$ converges and conclude that it is equal to $\pi/4$.

   The given series converges because 1) it is alternating, 2) its terms are strictly decreasing in magnitude, and 3) its terms have a limit of zero as $n \to \infty$, so the Alternating Series test applies.

   The limit of the series is $\pi/4$ since $\pi/4 = \arctan(1)$.

6. Prove that $\sum_{n=1}^{\infty} \cos(n)/n^2$ converges.

   $|\cos(n)/n^2| \leq 1/n^2$, so the series converges absolutely because $\sum_{n=1}^{\infty} 1/n^2$ converges.

7. Can you tell whether $\sum_{n=1}^{\infty} \cos(n)/n$ converges?

   Not without difficulty, since the series isn't alternating and the terms in the series aren't strictly decreasing in magnitude.