Midterm 2, Spring 2014–Solutions

1. For each of the following series determine whether the series is divergent, conditionally convergent, or absolutely convergent. Indicate which tests you used.

(a)

$$\sum_{n=1}^{\infty}(-1)^n \frac{1}{(n^2+1)^{1/3}}$$

The series is alternating, it has a limit of zero, and the terms are decreasing in magnitude, so by the Alternating Series test it converges.

Since $\lim_{n\to\infty} \frac{1}{(n^2+1)^{1/3}}(n^{2/3}) = 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges by the p-test, $\sum_{n=1}^{\infty} \frac{1}{(n^2+1)^{1/3}}$ diverges by the Limit Comparison Test (note: a direct comparison to $1/n^{2/3}$ will not work since $1/(n^2+1)^{1/3}$ is smaller).

The series is therefore convergent but not absolutely convergent, and so is conditionally convergent.

(b)

$$\sum_{n=1}^{\infty} \sin(n)\sqrt{\frac{n^2+n}{n^2+1}}$$

Since $\lim_{n\to\infty} \sqrt{\frac{n^2+n}{n^2+1}} = 1$ but $\lim_{n\to\infty} \sin(n)$ does not exist, $\lim_{n\to\infty} \sin(n)\sqrt{\frac{n^2+n}{n^2+1}}$ does not exist. The series therefore diverges by the Test For Divergence.

(c)

$$\sum_{n=6}^{\infty} \frac{\sin(n^2+2)}{n(n-5)}$$

Since $|\sin(x)| \leq 1$ for all $x \in \mathbb{R}$, it follows that $|\frac{\sin(n^2+2)}{n(n-5)}| \leq \frac{1}{n(n-5)}$.

Since $\lim_{n\to\infty} \frac{1}{n(n-5)}(n^2) = 1$, $\sum_{n=6}^{\infty} \frac{1}{n(n-5)}$ converges by the Limit Comparison Test. The series $\sum_{n=6}^{\infty} \frac{\sin(n^2+2)}{n(n-5)}$ is therefore absolutely convergent.

2. True/False. If true, explain why (concisely). If false, give a counterexample.

   (a) If the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} \sin(\ln(n))a_n$ converges.
   True. Since $|\sin(\ln(n))a_n| \leq |a_n|$ ($|\sin(x)| \leq 1$), the Convergence test implies that $\sum_{n=1}^{\infty} \sin(\ln(n))a_n$ also converges absolutely.

   (b) If $\{a_n\}$ converges but $\{b_n\}$ diverges, then $\{a_n b_n\}$ diverges.
   False: $a_n = 0$ is a (trivial) counterexample.

   Note: all counterexamples here will at least require $\lim_{n\to\infty} a_n = 0$.

   (c) If $\sum_{n=1}^{\infty} a_n$ converges absolutely and $\{b_n\}$ is bounded, then $\sum_{n=1}^{\infty} a_n b_n$ converges.
   True: If b_n is bounded then there is some M such that $|b_n| < M$ for all $n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} |a_n b_n| \leq \sum_{n=1}^{\infty} M \cdot |a_n| = M \sum_{n=1}^{\infty} |a_n|$, which converges. Therefore $\sum_{n=1}^{\infty} a_n b_n$ converges absolutely.
   Note: The condition that $\sum_{n=1}^{\infty} a_n$ converges *absolutely* is necessary here. Otherwise $a_n = (-1)^n/n, b_n = (-1)^n$ could serve as a counterexample.

   (d) If the series $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} a_n^2$ converges.
   False: $a_n = (-1)^n/\sqrt{n}$ serves as a counterexample.

   Note: If it were also specified that $a_n > 0$ for all $n \in \mathbb{N}$, then the statement would be true by the Comparison Test: $\lim_{n\to\infty} a_n = 0$ since $\sum_{n=1}^{\infty} a_n$ converges, so there exists an M such that $a_n < 1$ for all $n > M$, and therefore $a_n^2 < a_n$ for all $n > M$.

3. Find the first three non-zero terms of the Taylor series about $x = 0$ for

$$f(x) = (1 + x)\ln(1 + x^2)$$

Taking derivatives here would be a pain, so use the fact that you have the expansion for $\ln(1 + x)$ on your cheat sheet (hint, hint):

$$\begin{aligned} \ln(1+x) &= x - x^2/2 + x^3/3 - \dots \\ \ln(1+x^2) &= x^2 - x^4/2 + \dots \\ (1+x)\ln(1+x^2) &= x^2 + x^3 - x^4/2 - x^5/2 + \dots \end{aligned}$$

We know that we can cut the expansion for $\ln(1 + x^2)$ off at $x^2 - x^4/2$ because the next power will be x^6 but we only need the first three non-zero terms (i.e. x^2, x^3, x^4) for the answer.

4. Find the interval of convergence for the power series

$$\sum_{n=1}^{\infty} (-1)^n \frac{(2x-1)^n}{n}$$

Using the ratio test, we get

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{n|2x-1|^{n+1}}{(n+1)|2x-1|^n} \\
&= \lim_{n\to\infty} \frac{n}{n+1}|2x-1| \\
&= |2x-1|
\end{aligned}$$

Since the ratio test says that the series converges when $|r| < 1$, we know that the power series will converge when

$$\begin{aligned}
|2x-1| &< 1 \\
-1 < 2x-1 &< 1 \\
0 < 2x &< 2 \\
0 < x &< 1
\end{aligned}$$

Then plugging in $x = 0$, we get $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. Plugging in $x = 1$ gives $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges by the Alternating Series test. The interval of convergence is therefore $(0, 1]$. Check #1 to

confirm: when we plug in the numbers 0 and 1 for x, the exponential parts of the series become 1^n and $(-1)^n$, respectively. This will always be the case at the borders of the interval. Check #2 to confirm:

The exponential part (ignoring the $(-1)^n$) is $(2x-1)^n = 2^n(x - \frac{1}{2})^n$, so the center of convergence is $1/2$ and the radius is also $1/2$, giving an interval of $(0, 1)$. Since the non-exponential part is $\frac{1}{n}$ we expect exactly one of the two endpoints to converge.

5. True or False: You do not have to show your work.

(a) If $\sum_{n=1}^{\infty} a_n(x-1)^n$ converges at $x = 4$ and diverges at $x = -2$, then it converges at $x = -1$.

True. The interval of convergence has center $x = 1$ and radius at least 3 (since it converges at $x = 4$). -1 is within 3 of $x = 1$, and so the series converges at $x = -1$.

Note that since -2 is also distance 3 from $x = 1$, the interval of convergence must be $(-2, 4]$ precisely.

(b) If a series $\sum_{n=1}^{\infty} a_n 3^n$ converges, then $\sum_{n=1}^{\infty} a_n 2^n$ converges.

True. The Ratio Test combined with the fact that $\sum_{n=1}^{\infty} a_n 3^n$ converges implies that $\lim_{n\to\infty} 3|a_{n+1}/|a_n| \leq 1$. Applying the Ratio test to the second series then gives $\lim_{n\to\infty} 2|a_{n+1}|/|a_n| < 1$, which shows that it converges.

Also: one could think of the series $\sum_{n=1}^{\infty} a_n 3^n$ as the power series $a_n x^n$ (which has a center of 0) converging at $x = 3$. The power series therefore has a radius of at least 3, and so converges at $x = 2$.

(c) If the series $\sum_{n=1}^{\infty} a_n$ converges conditionally, then the radius of convergence of $\sum_{n=1}^{\infty} a_n(x-1)^n$ is 1.

True, since for $\sum_{n=1}^{\infty} a_n$ to converge conditionally it must look something like $1/n, 1/\ln n$, or $1/\sqrt{n}$. None of these grow or decay as fast as any exponential function (since then $\sum_{n=1}^{\infty} a_n$ would converge absolutely), and therefore these terms do not affect the radius of convergence.

This means that $\sum_{n=1}^{\infty} a_n(x-1)^n$ has the same radius of convergence as $\sum_{n=1}^{\infty}(x-1)^n$, which is 1.

(d) It is possible that the series $\sum_{n=1}^{\infty} a_n 3^n$ converges absolutely, but the series $\sum_{n=1}^{\infty} a_n(-2)^n$ diverges.

False: $|a_n(-2)^n| < |a_n 3^n|$, so if $\sum_{n=1}^{\infty} a_n 3^n$ converges absolutely then $\sum_{n=1}^{\infty} a_n 2^n$ does too.

See also the reasoning in part b).

(e) If the series $\sum_{n=1}^{\infty} a_n x^n$ has radius of convergence 1, then $\sum_{n=1}^{\infty} a_n/n^3$ converges.

False: $a_n = n^3$ serves as a counterexample. The point to keep in mind here is that if a_n is ANY sub-exponential function ($n^5, 1/n, \ln n + n$, etc.) then the series $\sum_{n=1}^{\infty} a_n x^n$ will have radius of convergence 1.