Midterm 1, Spring 2014–Solutions

1. Evaluate the integral

$$\int \frac{x^2+2x+4}{x+2}x^2+1\,dx$$

The rational function is proper, so we don't have to do long division and can instead go right to partial fractions:

$$\begin{aligned}\frac{x^2+2x+4}{x+2}x^2+1 &= \frac{A}{x+2}+\frac{Bx+C}{x^2+1}\\ x^2+2x+4 &= A(x^2+1)+(Bx+C)(x+2)\\ 1 &= A+B\\ 2 &= 2B+C\\ 4 &= A+2C\end{aligned}$$

Solving gives $A = 4/5, B = 1/5, C = 8/5$. This leaves us with

$$\begin{aligned}\int \frac{A}{x+2}+\frac{Bx+C}{x^2+1} &= A\ln(|x+2|)+\frac{B}{2}\ln(x^2+1)+C\arctan(x)\\ &= \frac{4}{5}\ln(|x+2|)+\frac{1}{10}\ln(x^2+1)+\frac{8}{5}\arctan(x)\end{aligned}$$

2. Evaluate the integral

$$\int x\sqrt{x^2-4x+5}\,dx$$

First complete the square to get

$$\int x\sqrt{x^2-4x+5}\,dx = \int x\sqrt{(x-2)^2+1}\,dx$$

This follows the $x = \tan\theta$ pattern, so substitute

$$(x-2) = \tan\theta, dx = \sec\theta\tan\theta\,d\theta$$

This turns the integral into

$$\begin{aligned}\int x\sqrt{(x-2)^2+1}\,dx &= \int (\tan\theta+2)\sqrt{\tan^2\theta+1}\sec^2\theta\,d\theta \\ &= \int \tan\theta\sec^3\theta + 2\sec^3\theta\,d\theta \\ &= \int \tan\theta\sec^3\theta\,d\theta + 2\int \sec^3\theta\,d\theta\end{aligned}$$

For the first integral, make the substitution $u = \sec\theta, du = \sec\theta\tan\theta$ to get

$$\begin{aligned}\int \tan\theta\sec^3\theta\,d\theta &= \int u^2\,du \\ &= \frac{1}{3}u^3 \\ &= \frac{1}{3}\sec^3\theta \\ &= \frac{1}{3}(x^2-4x+5)^{3/2},\end{aligned}$$

where the last bit comes from the fact that $\sqrt{x^2-4x+5}$ became $\sec\theta$ in the original integral (alternately, combine $(x-2) = \tan\theta$ with $\sec^2\theta = \tan^2\theta+1$). As for the second integral, use integration by parts with $u = \sec\theta, dv = \sec^2\theta\,d\theta$ to get

$$\begin{aligned}\int \sec^3\theta\,d\theta &= \sec\theta\tan\theta - \int \tan\theta(\sec\theta\tan\theta)\,d\theta \\ &= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta-1)\,d\theta \\ &= \sec\theta\tan\theta + \int \sec\theta\,d\theta - \int \sec^3\theta\,d\theta \\ 2\int \sec^3\theta\,d\theta &= \sec\theta\tan\theta + \ln|\sec\theta+\tan\theta| \\ &= (x-2)\sqrt{x^2-4x+5} + \ln|(x-2)+\sqrt{x^2-4x+5}|\end{aligned}$$

Therefore, putting this all together, we arrive at

$$\int x\sqrt{x^2-4x+5}\,dx = \frac{1}{3}(x^2-4x+5)^{3/2} + (x-2)\sqrt{x^2-4x+5} + \ln|(x-2)+\sqrt{x^2-4x+5}|$$

(HT GSI Alexander Bertoloni-Meli on figuring out how to integrate $\sec^3\theta$.)

3. (a) Indicate which of the following statements are true and which are false. Give a counterexample if false. DO NOT show your work if the statement is true.

   i. If $f(x) \geq 1$ and $\int_0^\infty x f(x)\,dx$ converges, then $\int_0^\infty f(x)\,dx$ also converges.
   This problem is broken, since if $f(x) \geq 1$ then $xf(x) \geq x$, and so $\int_0^\infty x f(x)\,dx \geq \int_0^\infty x\,dx$ cannot possibly converge. So instead assume the restriction $f(x) \geq 0$.
   The statement is false. $f(x) = e^{-x}/x$ and $f(x) = \begin{cases} 1/x & x < 1 \\ 0 & x \geq 1 \end{cases}$ both serve as counterexamples.

   ii. If $\int_{-1}^2 f(x)\,dx$ converges, then $\int_0^1 f(x)\,dx$ also converges.
   True. If an integral converges then the integral on every sub-interval of the original interval must also converge.

(b) Indicate which statements are true and which are false. You DO NOT HAVE TO show your work.

   i. $\displaystyle\int_1^\infty \frac{x+2}{x^{1/2}(1-x)^{1/2}}\,dx$ converges.
   This problem is also broken since $(1-x)^{1/2} = \sqrt{1-x}$ is not defined for $x > 1$. So assume it's $(x-1)^{1/2}$ instead.
   Comparing powers in the top and bottom gives $\frac{x}{x^{1/2}x^{1/2}} \sim 1$. The limit of the function is 1 as $x \to \infty$, and so the integral does not converge. FALSE.

   ii. $\displaystyle\int_0^\infty \frac{\sin(x)}{x^3}\,dx$ converges.
   FALSE. Don't let the ∞ fool you–it diverges because of what happens at 0. Peeling off a $\sin(x)/x$ (which approaches 1 as $x \to 0$), we get $\frac{\sin(x)}{x^3} \sim \frac{1}{x^2}$ as x approaches zero. Therefore the function grows like $1/x^2$ near zero, and so the integral diverges.

   iii. $\displaystyle\int_0^\pi \frac{\sin(x)-1}{x-\pi/2}$ converges.
   True. The potential problem is at $x = \pi/2$, but $\sin(\pi/2) - 1 = 0$. Using L'Hop's rule, we get

$$\begin{aligned} \lim_{x\to\pi/2} \frac{\sin(x)-1}{x-\pi/2} &= \lim_{x\to\pi/2} \frac{\cos(x)}{1} \\ &= 0 \end{aligned}$$

   So the function is not only bounded, it's limit is zero at $\pi/2$! So the integral definitely converges.

4. Evaluate the integral

$$\int \sin(\sqrt{x})\, dx$$

Substitute $y = \sqrt{x}, dy = 1/2\sqrt{x}\, dx = \frac{1}{2y}\, dx, 2ydy = dx$ to get

$$\int \sin(\sqrt{x})\, dx = 2 \int y \sin y\, dy$$

Then continue with integration by parts: $u = y, dv = \sin y$:

$$\begin{aligned} 2 \int y \sin y\, dy &= 2(-y \cos y - \int (-\cos y)\, dy) \\ &= -2y \cos y + 2 \sin y \\ &= -2\sqrt{x} \cos \sqrt{x} + 2 \sin \sqrt{x} \end{aligned}$$

5. Find the number of intervals n in the midpoint approximation so that the approximation of the integral

$$\int_0^1 \cos(x^2+1)\,dx$$

is accurate to within 10^{-4}. DO NOT COMPUTE THE APPROXIMATION.

First take the second derivative:

$$\begin{aligned}
f(x) &= \cos(x^2+1) \\
f'(x) &= -2x\sin(x^2+1) \\
f''(x) &= -2\sin(x^2+1) - 4x^2\cos(x^2+1) \\
|f''(x)| &\le |-2\sin(x^2+1) - 4x^2\cos(x^2+1)| \\
&\le 2|\sin(x^2+1)| + 4x^2|\cos(x^2+1)| \\
&\le 2 + 4x^2 \\
&\le 6
\end{aligned}$$

with the final inequality holding since $x \in [0,1]$. So 6 is an acceptable value for K. The from the error formula, we want

$$\begin{aligned}
\frac{K(b-a)^3}{24n^2} &\le 10^{-4} \\
\frac{10^4 K}{24} &\le n^2 \\
100\sqrt{K/24} &\le n \\
100\sqrt{6/24} &\le n \\
100/2 &\le n \\
50 &\le n
\end{aligned}$$

So $n \ge 50$ intervals will guarantee that the error in our approximation is at most 10^{-4}.