Midterm 1, Spring 2014–Solutions

1. Evaluate the integral

\[ \int \frac{x^2 + 2x + 4}{x+2} \frac{x^2 + 1}{dx} \]

The rational function is proper, so we don’t have to do long division and can instead go right to partial fractions:

\[ \frac{x^2 + 2x + 4}{x+2} = \frac{A}{x+2} + \frac{Bx+C}{x^2 + 1} \]

\[ x^2 + 2x + 4 = A(x^2 + 1) + (Bx + C)(x + 2) \]

\[ 1 = A + B \]
\[ 2 = 2B + C \]
\[ 4 = A + 2C \]

Solving gives \( A = \frac{4}{5}, B = \frac{1}{5}, C = \frac{8}{5} \). This leaves us with

\[ \int \frac{A}{x+2} + \frac{Bx+C}{x^2 + 1} = A \ln(|x + 2|) + B \ln(x^2 + 1) + C \arctan(x) \]

\[ = \frac{4}{5} \ln(|x + 2|) + \frac{1}{10} \ln(x^2 + 1) + \frac{8}{5} \arctan(x) \]
2. Evaluate the integral

\[ \int x\sqrt{x^2 - 4x + 5} \, dx \]

First complete the square to get

\[ \int x\sqrt{x^2 - 4x + 5} \, dx = \int x\sqrt{(x - 2)^2 + 1} \, dx \]

This follows the \( x = \tan \theta \) pattern, so substitute

\((x - 2) = \tan \theta, dx = \sec \theta \tan \theta \, d\theta\)

This turns the integral into

\[ \int x\sqrt{(x - 2)^2 + 1} \, dx = \int (\tan \theta + 2)\sqrt{\tan^2 \theta + 1} \, d\theta \]

\[ = \int \tan \theta \sec^3 \theta \, d\theta + 2 \int \sec^3 \theta \, d\theta \]

For the first integral, make the substitution \( u = \sec \theta, du = \sec \theta \tan \theta \, d\theta \) to get

\[ \int \tan \theta \sec^3 \theta \, d\theta = \int u^2 \, du \]

\[ = \frac{1}{3} u^3 \]

\[ = \frac{1}{3} \sec^3 \theta \]

\[ = \frac{1}{3} (x^2 - 4x + 5)^{3/2}, \]

where the last bit comes from the fact that \( \sqrt{x^2 - 4x + 5} \) became \( \sec \theta \) in the original integral (alternately, combine \((x - 2) = \tan \theta \) with \( \sec^2 \theta = \tan^2 \theta + 1 \)). As for the second integral, use integration by parts with \( u = \sec \theta, dv = \sec^2 \theta \, d\theta \) to get

\[ \int \sec^3 \theta \, d\theta = \sec \theta \tan \theta - \int \sec \theta (\sec \theta \tan \theta) \, d\theta \]

\[ = \sec \theta \tan \theta - \int \sec \theta (\sec^2 \theta - 1) \, d\theta \]

\[ = \sec \theta \tan \theta + \int \sec \theta \, d\theta - \int \sec^3 \theta \, d\theta \]

\[ 2 \int \sec^3 \theta \, d\theta = \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \]

\[ = (x - 2)\sqrt{x^2 - 4x + 5} + \ln |(x - 2) + \sqrt{x^2 - 4x + 5}| \]

Therefore, putting this all together, we arrive at

\[ \int x\sqrt{x^2 - 4x + 5} \, dx = \frac{1}{3} (x^2 - 4x + 5)^{3/2} + (x - 2)\sqrt{x^2 - 4x + 5} + \ln |(x - 2) + \sqrt{x^2 - 4x + 5}| \]

(HT GSI Alexander Bertoloni-Meli on figuring out how to integrate \( \sec^3 \theta \).)
3. (a) Indicate which of the following statements are true and which are false. Give a counterexample if false. DO NOT show your work if the statement is true.

i. If \( f(x) \geq 1 \) and \( \int_0^\infty x f(x) \, dx \) converges, then \( \int_0^\infty f(x) \, dx \) also converges.
   This problem is broken, since if \( f(x) \geq 1 \) then \( x f(x) \geq x \), and so \( \int_0^\infty x f(x) \, dx \geq \int_0^\infty x \, dx \) cannot possibly converge. So instead assume the restriction \( f(x) \geq 0 \).
   The statement is false. \( f(x) = e^{-x}/x \) and \( f(x) = \begin{cases} 1/x & x < 1 \\ 0 & x \geq 1 \end{cases} \) both serve as counterexamples.

ii. If \( \int_2^{-1} f(x) \, dx \) converges, then \( \int_0^1 f(x) \, dx \) also converges.
   True. If an integral converges then the integral on every sub-interval of the original interval must also converge.

(b) Indicate which statements are true and which are false. You DO NOT HAVE TO show your work.

i. \( \int_1^\infty \frac{x + 2}{x^{1/2}(1 - x)^{1/2}} \, dx \) converges.
   This problem is also broken since \( (1 - x)^{1/2} = \sqrt{1 - x} \) is not defined for \( x > 1 \). So assume it’s \( (x - 1)^{1/2} \) instead.
   Comparing powers in the top and bottom gives \( \frac{x}{(x - 1)^{1/2}} \sim 1 \). The limit of the function is 1 as \( x \to \infty \), and so the integral does not converge. FALSE.

ii. \( \int_0^\infty \frac{\sin(x)}{x^3} \, dx \) converges.
   FALSE. Don’t let the \( \infty \) fool you—it diverges because of what happens at 0. Peeling off a \( \sin(x)/x \) (which approaches 1 as \( x \to 0 \)), we get \( \frac{\sin(x)}{x} \sim \frac{1}{x} \) as \( x \) approaches zero. Therefore the function grows like \( 1/x^2 \) near zero, and so the integral diverges.

iii. \( \int_0^\pi \frac{\sin(x) - 1}{x - \pi/2} \) converges.
   True. The potential problem is at \( x = \pi/2 \), but \( \sin(\pi/2) - 1 = 0 \). Using L’Hop’s rule, we get
   \[
   \lim_{x \to \pi/2} \frac{\sin(x) - 1}{x - \pi/2} = \lim_{x \to \pi/2} \frac{\cos(x)}{1} = 0
   \]
   So the function is not only bounded, it’s limit is zero at \( \pi/2 \)! So the integral definitely converges.
4. Evaluate the integral 
\[ \int \sin(\sqrt{x}) \, dx \]

Substitute \( y = \sqrt{x}, \, dy = \frac{1}{2\sqrt{x}} \, dx = \frac{1}{2y} \, dx \), \( 2y \, dy = dx \) to get
\[ \int \sin(\sqrt{x}) \, dx = 2 \int y \sin y \, dy \]

Then continue with integration by parts: \( u = y, \, dv = \sin y \):
\[
2 \int y \sin y \, dy = 2( -y \cos y - \int ( -\cos y ) \, dy )
\]
\[
= -2y \cos y + 2 \sin y
\]
\[
= -2\sqrt{x} \cos \sqrt{x} + 2 \sin \sqrt{x}
\]
5. Find the number of intervals \( n \) in the midpoint approximation so that the approximation of the integral
\[
\int_0^1 \cos(x^2 + 1) \, dx
\]
is accurate to within \( 10^{-4} \). DO NOT COMPUTE THE APPROXIMATION.

First take the second derivative:

\[
\begin{align*}
f(x) &= \cos(x^2 + 1) \\
f'(x) &= -2x \sin(x^2 + 1) \\
f''(x) &= -2 \sin(x^2 + 1) - 4x^2 \cos(x^2 + 1) \\
|f''(x)| &\leq | -2 \sin(x^2 + 1) - 4x^2 \cos(x^2 + 1) | \\
&\leq 2 | \sin(x^2 + 1) | + 4x^2 | \cos(x^2 + 1) | \\
&\leq 2 + 4x^2 \\
&\leq 6
\end{align*}
\]

with the final inequality holding since \( x \in [0, 1] \). So 6 is an acceptable value for \( K \). The from the error formula, we want

\[
\frac{K(b - a)^3}{24n^2} \leq 10^{-4}
\]

\[
\frac{10^4K}{24} \leq n^2
\]

\[
100\sqrt{K/24} \leq n
\]

\[
100\sqrt{6/24} \leq n
\]

\[
50 \leq n
\]

So \( n \geq 50 \) intervals will guarantee that the error in our approximation is at most \( 10^{-4} \).