Solutions to 2014 Final

1. Evaluate the integral $\int \sin(2x)e^x\,dx$

   Integration by parts. First choose $u = \sin 2x, dv = e^x\,dx$ and get $du = 2\cos 2x, v = e^x$. In a second round, choose $u = \cos 2x$ and $dv = e^x\,dx$. After two rounds of integration by parts the original integral appears on the right-hand side, so move it over to the left and solve.

$$\begin{aligned}
\int \sin(2x)e^x\,dx &= e^x \sin 2x - 2\int \cos 2x e^x\,dx \\
&= e^x \sin 2x - 2\left(e^x \cos 2x - 2\int -\sin 2x e^x\,dx\right) \\
&= e^x \sin 2x - 2e^x \cos 2x - 4\int \sin 2x e^x\,dx \\
5\int \sin(2x)e^x\,dx &= e^x \sin(2x) - 2e^x \cos(2x) \\
\int \sin(2x)e^x\,dx &= \frac{1}{5}e^x \sin(2x) - \frac{2}{5}e^x \cos(2x)
\end{aligned}$$

   But wait! Now that we've done differential equations, we have a new way to solve the problem! This is the same as solving $y' = \sin(2x)e^x$, so we can use undetermined coefficients and guess

$$\begin{aligned}
y &= A\sin(2x)e^x + B\cos(2x)e^x \\
y' &= A\sin(2x)e^x + 2A\cos(2x)e^x + B\cos(2x)e^x - 2B\sin(2x)e^x \\
A - 2B &= 1 \\
2A + B &= 0 \\
A &= \frac{1}{5} \\
B &= \frac{-2}{5}
\end{aligned}$$

   But wait again! Now that we've covered complex numbers, we have yet another way! Use the identities

$$e^{i\theta} = \cos\theta + i\sin\theta, \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\begin{aligned}
\int \sin(2x)e^x\,dx &= \frac{1}{2i}\int (e^{2ix} - e^{-2ix})e^x\,dx \\
&= \frac{1}{2i}\int e^{(1+2i)x} - e^{(1-2i)x}\,dx \\
&= \frac{1}{2i}\left(\frac{1}{1+2i}e^{(1+2i)x} - \frac{1}{1-2i}e^{(1-2i)x}\right) \\
&= \frac{e^x}{2i}\left(\frac{1}{1+2i}e^{2ix} - \frac{1}{1-2i}e^{-2ix}\right) \\
&= \frac{e^x}{2i}\left(\frac{1-2i}{5}(\cos 2x + i\sin 2x) - \frac{1+2i}{5}(\cos 2x - i\sin 2x)\right) \\
&= \frac{e^x}{2i}\left(\frac{-4i}{5}\cos 2x + \frac{2i}{5}\sin 2x\right) \\
&= \frac{-2}{5}e^x \cos 2x + \frac{1}{5}e^x \sin 2x
\end{aligned}$$

. . . okay, maybe don't solve it that last way. But using undetermined coefficients can be useful!

2. Compute the integral $\int \frac{1}{(t-1)(t^2+1)}\,dt$

First find the partial fraction decomposition:

$$\begin{aligned}
\frac{1}{(t-1)(t^2+1)} &= \frac{A}{t-1} + \frac{Bt+C}{t^2+1} \\
1 &= A(t^2+1) + (Bt+C)(t-1) \\
0 \cdot t^2 &= (A+B)t^2 \\
0 \cdot t &= (B-C)t \\
1 &= A - C
\end{aligned}$$

Solving the three equations above gives $A = \frac{1}{2}, B = C = \frac{-1}{2}$. Therefore

$$\begin{aligned}
\frac{1}{(t-1)(t^2+1)} &= \frac{1}{2}\left(\frac{1}{t-1} - \frac{t+1}{t^2+1}\right) \\
\int \frac{1}{(t-1)(t^2+1)} &= \frac{1}{2}\int \frac{1}{t-1} - \frac{1}{2}\int \frac{t+1}{t^2+1} \\
&= \frac{1}{2}\ln|t-1| - \frac{1}{4}\ln(t^2+1) - \frac{1}{2}\arctan t
\end{aligned}$$

3. Decide whether each improper integral is convergent or divergent. Do not show your work.

(a) $\int_1^\infty \frac{dx}{x \ln x}$

Since $\int \frac{dx}{x \ln x} = \ln \ln x$, and $\lim_{x\to\infty} \ln \ln x = \infty$, the integral must be DIVERGENT. The integral is also divergent because of the vertical asymptote at $x = 1$, since $\lim_{x\to 1} \ln \ln x = -\infty$.

(b) $\int \frac{dx}{x(\ln x)^2}$

$$\begin{aligned}\int_1^\infty \frac{dx}{x(\ln x)^2} &= \lim_{a\to 1^+} \lim_{b\to\infty} \frac{-1}{\ln x}\Big|_a^b \\ &= \lim_{a\to 1^+} \lim_{b\to\infty} \frac{1}{\ln a} - \frac{1}{\ln b} \\ &= \lim_{a\to 1+} \frac{1}{\ln a} \\ &= \infty\end{aligned}$$

Because of the vertical asymptote at $x = 1$, the integral is DIVERGENT.

(c) $\int_{\pi/2}^\infty \frac{(\cos x)^2}{x - \frac{\pi}{2}} dx$

The fact that the denominator is zero at $x = \pi/2$ is not problematic because the numerator has a double zero at that same point. However, note that

$$\int \frac{\cos^2 x}{x - \pi/2} + \frac{\sin^2 x}{x - \pi/2} = \int \frac{1}{x - \pi/2},$$

which diverges on the interval $[\pi, \infty]$ by the p-test. At least one of the two integrals in the sum is therefore divergent, and it is most likely that they both are. Thus DIVERGENT, though the formal proof is somewhat harder.

(d) $\int_0^\infty \frac{\sin x^2}{x^2} dx$

The point $x = 0$ is not a problem since $\lim_{x\to 0} \frac{\sin x^2}{x^2} = 1$, and the limit as $x \to \infty$ is not a problem since $\frac{\sin x^2}{x^2} \leq \frac{1}{x^2}$, which converges. The integral is therefore CONVERGENT.

(e) $\int_0^1 \frac{dx}{x\sqrt{1-x}}$

DIVERGENT because of the $\frac{1}{x}$ term at the point $x = 0$. More formally, since $\sqrt{1-x} \leq 1$ for $x \in [0, 1]$, we have

$$\int_0^1 \frac{dx}{x\sqrt{1-x}} \geq \int_0^1 \frac{dx}{x}.$$

4. Find the radius and interval of convergence of the power series $\sum_{n=0}^{\infty}(-1)^n \frac{1}{\sqrt{n+5}}(x-2)^n$.

At a glance: the term $\frac{1}{\sqrt{n+5}}$ is fluff, so ignore it. The center is at $x = 2$ and the radius is 1 so the interval of convergence is $(1, 3)$ (up to the endpoints). The series is alternating when $x = 3$ but looks like $\frac{1}{\sqrt{n+5}}$ when $x = 1$, so the interval of convergence is $(1, 3]$.

More formally for the interval of convergence:

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{\sqrt{n+5}}{\sqrt{n+6}} \frac{|x-2|^{n+1}}{|x-2|^n} \\
&= \lim_{n\to\infty} \frac{\sqrt{n+5}}{\sqrt{n+6}} |x-2| \\
&= |x-2|
\end{aligned}$$

The ratio test implies that the series converges whenever this quantity is less than 1, so it converges whenever

$$\begin{aligned}
|x-2| &< 1 \\
x &\in (2-1, 2+1) \\
x &\in (1, 3)
\end{aligned}$$

5. State whether each of the following series is absolutely convergent, conditionally convergent, or divergent. You do not have to show your work.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$

$$\begin{aligned}(-1)^n \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} &= (-1)^n \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \\ &= (-1)^n \frac{1}{(\sqrt{n+1}+\sqrt{n})^2}\end{aligned}$$

The series converges by the Alternating Series Test. It does not converge absolutely because

$$\begin{aligned}\frac{1}{(\sqrt{n+1}+\sqrt{n})^2} &\leq \frac{1}{(2\sqrt{n})^2} \\ &= \frac{1}{4n},\end{aligned}$$

which is a multiple of the Harmonic Series and so diverges (or, Integral Test with 1/x). The series is therefore CONDITIONALLY CONVERGENT.

(b) $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{\pi}{2}+\frac{1}{n^2}\right)$

ABSOLUTELY CONVERGENT, since $|\cos(\pi/2+1/n^2)| = |\sin(1/n^2)| < |1/n^2|$.

(c) $\displaystyle\sum_{n=1}^{\infty}\left|\frac{n\sin n}{n^3+1}\right| < \sum_{n=1}^{\infty}\frac{n}{n^3+1} < \sum_{n=1}^{\infty}\frac{1}{n^2}$, so the series is ABSOLUTELY CONVERGENT.

(d) $\displaystyle\sum_{n=1}^{\infty} \tan(1/n)\sin(\pi/2+\pi n)$

First, $\sin(\pi/2+\pi n) = (-1)^n$, and since $\lim_{n\to\infty}\tan(1/n) = 0$ (and $\tan 1/n > 0$) the series converges by the Alternating Series Test.

Second $\lim_{n\to\infty}\frac{\tan(1/n)}{1/n} = \lim_{n\to\infty}\frac{n\sin(1/n)}{\cos(1/n)} = 1$, so by the Limit Comparison Test with $1/n$ the series does not converge absolutely. Thus CONDITONALLY CONVERGENT.

(e) $\displaystyle\sum_{n=1}^{\infty} n^2\cos(\pi n/2)$

The terms in the series do not even have a limit of zero, so the series DIVERGES.

6. If True, give an explanation. If False, give a counterexample.

(a) If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} a_n^2$ diverges then $\sum_{n=1}^{\infty} a_n$ converges conditionally.
True, since if $\sum_{n=1}^{\infty} a_n$ converged absolutley we could use the comparison test with $0 < a_n^2 < |a_n|$ to show that $\sum_{n=1}^{\infty} a_n^2$ converged. But since $\sum_{n=1}^{\infty} a_n^2$ diverges, $\sum_{n=1}^{\infty} a_n$ cannot converge absolutely.

(b) If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} a_n + |a_n|$ converges.
False: $a_n = (-1)^n/n$.

(c) If $\sum_{n=1}^{\infty} a_n$ converges and $\lim_{n\to\infty} b_n$ exists and $b_n \neq 0$, then $\sum_{n=1}^{\infty} a_n b_n$ converges.
False: $a_n = b_n = (-1)^n/\sqrt{n}$.

7. True or False: You do not have to show your work.

(a) If $\sum_{n=1}^{\infty}(-1)^n a_n$ diverges then the radius of convergence R of $a_n(x-1)^n$ is less than or equal to 2.

True. The power series is centered at $x = 1$ but diverges at $x = 0$, so $R \leq 1$ (and therefore $R \leq 2$).

(b) If $\sum_{n=1}^{\infty} c_n(x-1)^n$ converges at $x = 3$ then $\sum_{n=1}^{\infty} c_n$ converges.

The series is centered at $x = 1$ and converges at $x = 3$, so it must also converge at $x = 2$ (where $\sum_{n=1}^{\infty} c_n(x-1)^n = \sum_{n=1}^{\infty} c_n$).

(c) The radius of convergence of $\sum_{n=1}^{\infty}(1+5^n)x^n$ is greater than 4.

False: the radius of converence is 1/5. (It is also simple to check that the series is centered at $x = 0$ but diverges at $x = 1$).

(d) Even though the series $\sum_{n=1}^{\infty} c_n(x-1)^n$ converges at $x = -1$, the series $\sum_{n=1}^{\infty} c_n 2^n$ may diverge.

True: the power series is centered at 1, and $\sum_{n=1}^{\infty} c_n 2^n$ corresponds to $x = 3$, and it is possible for the interval of convergence to be $[-1, 3)$.

(e) If the series $\sum_{n=1}^{\infty} c_n x^n$ converges absolutely for $|x| \leq 2$ then the radius of convergence is 2.

If it is implied that the bound $|x| \leq 2$ is tight (so $\sum_{n=1}^{\infty} c_n x^n$ does NOT converge absolutely for any $|x| > 2$), then true. Otherwise a sequence like $c_n = 0$ converges on $|x| \leq 2$ and has an infinite radius of convergence.

8. Find the general solution to the differential equation $yy' - y^2x = x$.

   This is not a linear differential equation, so our only hope of solving the problem is if the equation is separable. So let's try that:

$$\begin{aligned}
yy' - y^2x &= x \\
yy' &= x + y^2x \\
\frac{y}{1+y^2}\,dy &= x\,dx \\
\int \frac{y}{1+y^2}\,dy &= \int x\,dx \\
\frac{1}{2}\ln(1+y^2) &= \frac{1}{2}x^2 + C \\
\ln(1+y^2) &= x^2 + C \\
1 + y^2 &= e^{x^2+C} \\
y^2 &= e^{x^2+C} - 1 \\
y &= \sqrt{e^{x^2+C} - 1}
\end{aligned}$$

9. Find the solution to the initial value problem

$$y'' + y = x^2 + e^x, y(0) = -3/2, y'(0) = 1/2$$

First find a solution to $y'' + y = x^2$ by guessing $y = Ax^2 + Bx + C$ and get $y = x^2 - 2$ as a particular solution. Then find a solution to $y'' + y = e^x$ by guessing $y = Ke^x$ and get $y = \frac{1}{2}e^x$.

Then solve the homogeneous equation $y'' + y = 0$ by finding the roots to $r^2 + 1 = 0$. Get $r = \pm i$, which leads to $y = C_1 \cos x + C_2 \sin x$.

The general solution is therefore $y = x^2 + \frac{1}{2}e^x - 2 + C_1 \cos x + C_2 \sin x$. Adding in the constraint $y(0) = -3/2$ gives $C_1 = 0$, and the constraint $y'(0) = 1/2$ then gives $C_2 = 0$.

Therefore, the solution is $y = x^2 + \frac{1}{2}e^x - 2$.

10. Find the solution to the initial value problem

$$y' + y \tan x = \sec x, y(0) = 0$$

This problem is in the form $y' + P(x)y = Q(x)$, so use the integrating factor:

$$\begin{aligned}
I &= e^{\int \tan x} \\
&= e^{-\ln \cos x} \\
&= \frac{1}{\cos x} \\
y &= (\int IQ)/I \\
&= \cos x(\int \sec^2 x \, dx) \\
&= \cos x(\tan x + C) \\
&= \sin x + C \cos x
\end{aligned}$$

Then the initial condition gives $y = \sin x$. This is a very simple solution, so let's verify that it is in fact the correct solution:

$$\begin{aligned}
(\sin x)' + (\tan x) \sin x &=^? \sec x \\
\cos x + \frac{\sin^2 x}{\cos x} &=^? \frac{1}{\cos x} \\
\frac{\cos^2 x + \sin^2 x}{\cos x} &=^? \frac{1}{\cos x} \\
\cos^2 x + \sin^2 x &= 1
\end{aligned}$$

Yup!

11. Match pictures to differential equations. Let's start by finding the solutions to $\frac{dy}{dx} = 0$ in each case and seeing how they differ.

(a) $\frac{dy}{dx} = y^3 - x^3$

$y' = 0$ precisely when $y^3 = x^3$, which happens when $y = x$. If $y > x$ then $y' > 0$, and if $y < x$ then $y' < 0$. This is either graph 4 or 5... we'll need more information to decide.

(b) $y' = y^2/x^2$

$y' = 0$ only when $y = 0$, y' will be very large when $x \approx 0$, and y' is never negative. This does not describe any of the graphs accurately. The answer was presumably supposed to be GRAPH 1, which would be closer to $y' = y/x$.

(c) $y' = -x + y$

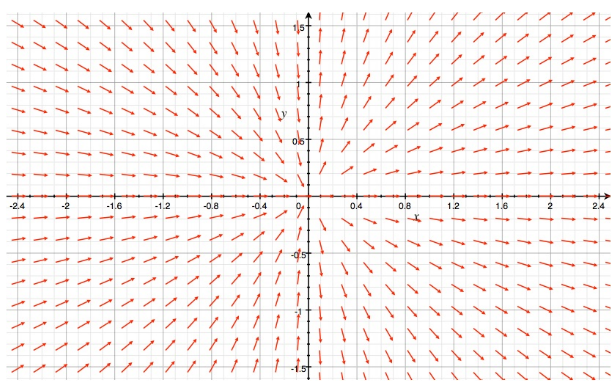
$y' = 0$ precisely when $y = x$. This is unfortunately the same condition as for (a), so we can also get $y' = 1$ when $y = x + 1$... another straight line, which won't be the case for (a). We can additionally separate this equation from (a) because in (a), y' will be much smaller when x and y are near 0, then grow faster when $y > 1$ or $x >$. This equation therefore describes GRAPH 5, and (a) describes GRAPH 4.

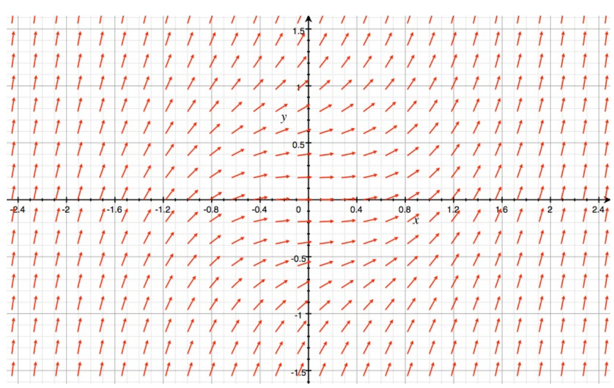
(d) $y' = x^2 + y^2$

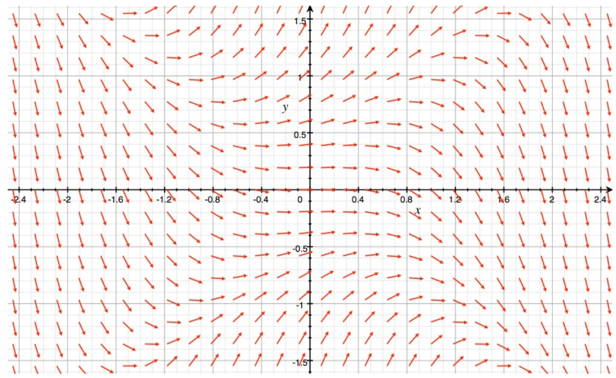
$y' = 0$ only at $(0, 0)$ and y' is positive everywhere else. GRAPH 2.

(e) $y' = y^2 - x^2$

$y' = 0$ precisely when $y = \pm x$. GRAPH 3.







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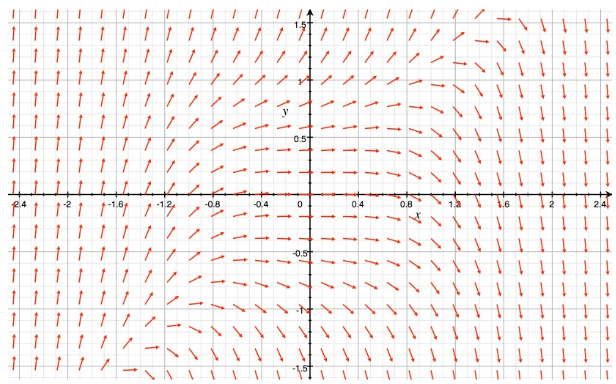

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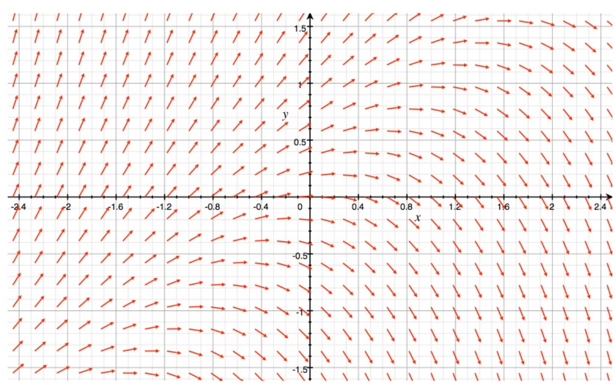

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