6.5+6.7: Least Squares

Tuesday, October 18

Warmup

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, express $\mathbf{u} \cdot \mathbf{v}$ in terms of matrix algebra (i.e. without using the "dot" symbol).

If $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$, give an interpretation in words of the matrix-vector products $A\mathbf{x}$ and $A^T\mathbf{x}$.

Least Squares

Suppose that $A = QR$ where R is invertible and Q is orthogonal. Show that $(A^T A)^{-1} A^T \mathbf{b} = R^{-1} Q^T \mathbf{b}$.

Show that $A(A^T A)^{-1} A^T \mathbf{b} = \hat{\mathbf{b}}$, where $\hat{\mathbf{b}} = \operatorname{proj}_A \mathbf{b}$.

Find an expression for the distance from $\mathbf{b}$ to the span of A in terms of $\mathbf{b}$, and A or $\mathbf{b}$ and Q.

More Least Squares

Say we want to find the best least-squares linear approximation $y = mx + b$ for the four points (-6,-1), (-2,2), (1,1), and (7,6). This amounts to solving the least-squares problem

$$\begin{bmatrix} b \\ m \end{bmatrix} = \arg\min_{\mathbf{x}} \left\| \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix} \right\|^2 .$$

Find the least-squares solution using either the normal equations or a QR decomposition, and plot the line and points.

Inner Product Spaces

Define an inner product on $\mathbb{P}_2$ by $\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$. Find the inner products between the vectors $1, t$, and t^2. Find an orthogonal basis for $\mathbb{P}_2$ with respect to this inner product.