6.7: Inner Product Spaces

Thursday, October 20

Orthogonality

Find a non-zero vector orthogonal to the vector $\begin{bmatrix} a \\ b \end{bmatrix}$. Then find *all* vectors orthogonal to $\begin{bmatrix} a \\ b \end{bmatrix}$.

Which of the following matrices are orthogonal?

$$\begin{bmatrix} 1 & -\sqrt{2}/2 \\ 0 & \sqrt{2}2 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

Suppose that Q is a (2×2) orthogonal matrix. Sketch the set of all points where Q might possibly take the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Make a few sketches for possible locations of $Q \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $Q \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Describe the effect of Q in words.

True/False

1. A least-squares solution of $A\mathbf{x} = \mathbf{b}$ is a vector $\widehat{\mathbf{x}}$ such that $A\widehat{\mathbf{x}} = \widehat{\mathbf{b}}$, where $\widehat{\mathbf{b}}$ is the projection of $\mathbf{b}$ onto the column space of A.
2. A least-squares solution of $A\mathbf{x} = \mathbf{b}$ is a vector $\widehat{\mathbf{x}}$ such that $\|\mathbf{b} - A\mathbf{x}\| \leq \|\mathbf{b} - A\widehat{\mathbf{x}}\|$ for all $\mathbf{x} \in \mathbb{R}^n$.
3. If the columns of A are linearly independent then the equation $A\mathbf{x} = \mathbf{b}$ has exactly one least-squares solution.
4. If $A = QR$ and Q has orthogonal columns, then $R = Q^T A$.
5. If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for V then so is $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\}$.

Inner Product Spaces

If $P((x_1, x_2), (y_1, y_2)) = x_1 y_1 + x_1 y_2$, is P an inner product? Why or why not?

Define an inner product on $\mathbb{P}_2$ by $\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$. Find the inner products between the vectors $1, t$, and t^2. Find an orthogonal basis for $\mathbb{P}_2$ with respect to this inner product.

If we define an inner product on $C[-\pi, \pi]$ by $\langle f, g \rangle = \frac{1}{2\pi} \int_{x=-\pi}^{\pi} f(x)g(x)\, dx$, find $\|\sin(x)\|$, $\|\sin(2x)\|$, and $\langle \sin(x), \sin(2x) \rangle$.