6.7: Inner Product Spaces

Thursday, October 20

Orthogonality

Find a non-zero vector orthogonal to the vector $\begin{bmatrix} a \\ b \end{bmatrix}$. Then find *all* vectors orthogonal to $\begin{bmatrix} a \\ b \end{bmatrix}$.

ANSWER: The set of all orthogonal vectors is $c \begin{bmatrix} -b \\ a \end{bmatrix}$ for any $c \in \mathbb{R}$.

Which of the following matrices are orthogonal?

$$\begin{bmatrix} 1 & -\sqrt{2}/2 \\ 0 & \sqrt{2}2 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

ANSWER: only the third one is orthogonal. The first has unit vector columns but they are not orthogonal, and the columns of the second are orthogonal to each other but are not unit vectors.

Suppose that Q is a (2×2) orthogonal matrix. Sketch the set of all points where Q might possibly take the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Make a few sketches for possible locations of $Q \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $Q \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Describe the effect of Q in words.
ANSWER: $Q\mathbf{e}_1$ must remain on the unit circle, and once $Q\mathbf{e}_1$ is fixed there are only two possible spots for $Q\mathbf{e}_2$, as it must form a right angle with $Q\mathbf{e}_1$. All such orthogonal matrices define a rigid rotation of the plane, and possibly with a reflection.

True/False

1. A least-squares solution of $A\mathbf{x} = \mathbf{b}$ is a vector $\widehat{\mathbf{x}}$ such that $A\widehat{\mathbf{x}} = \widehat{\mathbf{b}}$, where $\widehat{\mathbf{b}}$ is the projection of $\mathbf{b}$ onto the column space of A. TRUE.

2. A least-squares solution of $A\mathbf{x} = \mathbf{b}$ is a vector $\widehat{\mathbf{x}}$ such that $\|\mathbf{b} - A\mathbf{x}\| \leq \|\mathbf{b} - A\widehat{\mathbf{x}}\|$ for all $\mathbf{x} \in \mathbb{R}^n$. FALSE: the inequality should be flipped since we want the residual to be as small as possible.

3. If the columns of A are linearly independent then the equation $A\mathbf{x} = \mathbf{b}$ has exactly one least-squares solution. TRUE, since if $A = QR$ and A has linearly independent columns then R will be invertible.

4. If $A = QR$ and Q has orthogonal columns, then $R = Q^T A$. FALSE: Q must have *orthonormal* columns.

5. If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for V then so is $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\}$. FALSE: $(\mathbf{v}_1 + \mathbf{v}_2)^T(\mathbf{v}_1 - \mathbf{v}_2) = \|\mathbf{v}_1\|^2 - \|\mathbf{v}_2\|^2$, which will be zero if and only if $\|\mathbf{v}_1\| = \|\mathbf{v}_2\|$.

Inner Product Spaces

If $P((x_1, x_2), (y_1, y_2)) = x_1y_1 + x_1y_2$, is P an inner product? Why or why not?
ANSWER: Check the four possible properties one at a time:

- $\langle u, v\rangle = \langle v, u\rangle$: NO
- $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$: YES
- $\langle c\mathbf{u}, \mathbf{v}\rangle = c\langle \mathbf{u}, \mathbf{v}\rangle$: YES
- $\langle \mathbf{u}, \mathbf{u}\rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u}\rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$: NO.

Define an inner product on $\mathbb{P}_2$ by $\langle p, q\rangle = p(-1)q(-1)+p(0)q(0)+p(1)q(1)$. Find the inner products between the vectors $1, t$, and t^2. Find an orthogonal basis for $\mathbb{P}_2$ with respect to this inner product.
ANSWER: $\langle 1, t\rangle = \langle t, t^2\rangle = 0$. $\langle 1, t^2\rangle = 2$.
Since $\mathbf{1}$ and t^2 are both orthononal to t any linear combination of them will also be orthongal to t, so to find an orthogonal basis we can pick $\mathbf{1}$, t, and $t^2 + c \cdot \mathbf{1}$ for some constant c such that $0 = \langle \mathbf{1}, t^2 + c \cdot \mathbf{1}\rangle = \langle \mathbf{1}, t^2\rangle + c\|\mathbf{1}\|^2 = 2 + 3c$. Thus $c = -2/3$ will work.

If we define an inner product on $C[-\pi, \pi]$ by $\langle f, g\rangle = \frac{1}{2\pi}\int_{x=-\pi}^{\pi} f(x)g(x)\,dx$, find $\|\sin(x)\|, \|\sin(2x)\|$, and $\langle \sin(x), \sin(2x)\rangle$.
ANSWER: You will need some trig identities from caclulus to solve this one, but the answer will be that $\|\sin x\| = \|\sin 2x\| = 1$ and $\sin x, \sin 2x$ are orthogonal with respect to this inner product. (In fact, *all* functions of the form $\sin nx$ and $\cos nx$ are orthogonal with respect to this inner product).